Exam 23.2.2023

In each task, you can get a maximum of 6 points for the correct solution. You can reach a maximum of 30 points in the exam.

1. Explain shortly.
   a. What are Weierstrass-Erdmann corner point conditions?
   b. What is the cost-to-go function (DP)?
   c. What is the difference of an open-loop and a closed-loop control?
   d. What is Pontryagin's minimum principle?
   e.–f. What are the Euler-Lagrange equations and how are they derived?

2. Minimize the functional

$$J = \int_0^2 [\dot{x}^2(t) + 2x(t)\dot{x}(t) + 4x^2(t)]\, dt,$$

   using the calculus of variations, where $x(0) = 1$ and $x(2)$ is free.
   Explain the steps that you are taking.

3. Solve the minimization problem

$$J = \frac{a}{2}t_f^2 + \frac{b}{2}\int_0^{t_f} u^2(t)\, dt,$$

   where $a, b > 0$ are constants,

$$\ddot{x} = u,$$

   and $x(0) = 10$, $\dot{x}(0) = 0$, $x(t_f) = 0$, $\dot{x}(t_f) = -1$, and the terminal time $t_f > 0$ is free.
   Determine the optimal control u^*, the optimal state trajectory x^*, and the terminal time t_f in dependence of a and b.
   Hint: The state variable is $[x_1, x_2]^T$, where $x_1 = x$ and $x_2 = \dot{x}$.

Please turn! ↪

4. Let us examine the problem

$$\dot{x}(t) = -x(t) + u(t),\ x(0) = 1,\ x(1) = 0,$$

$$J = \int_0^1 \left[x^2(t) + u^2(t)\right]\, dt.$$

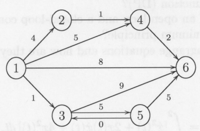
Solve the optimal control using the HJB. (No need to solve the differential equations).

Hint: Use the trial

$$V(t,x) = a(t)x^2 + 2b(t)x + c(t),$$

and derive differential equations for the functions a, b, c.

5. Find a shortest path from each node to node 6 for the graph below using the DP algorithm.



Appendix

HJB: $0 = J_t + \min_{u(t)} \left\{g + J_x^T f\right\}$

E-L: $0 = g_x - \frac{d}{dt}(g_{\dot{x}})$

Hamiltonian: $H(x(t), u(t), t) = g(x(t), u(t), t) + p^T(t) f(x(t), u(t), t)$

costate: $\dot{p}(t) = -\frac{\partial H}{\partial x}(\cdot, t)$

free final state: $0 = g_{\dot{x}}(\cdot, t_f)$ or $\ell_x(x(t_f), t_f) - p(t_f) = 0$

free final time: $0 = g(\cdot, t_f) - g_{\dot{x}}\dot{x}(\cdot, t_f)$ or $H(\cdot, t_f) + \ell_t(x(t_f), t_f) = 0$

free final state and time: $g(\cdot, t_f) = g_{\dot{x}}(\cdot, t_f) = 0$ or $\ell_x(x(t_f), t_f) - p(t_f) = 0 = H(\cdot, t_f) + \ell_t(x(t_f), t_f)$

goal: $0 = g(\cdot, t_f) + g_{\dot{x}}(\cdot, t_f)^T \left[\frac{d\theta(t_f)}{dt} - \dot{x}(t_f)\right]$ or $H(\cdot, t_f) + \ell_t(x(t_f), t_f) + (\ell_x(x(t_f), t_f) - p(t_f))^T \frac{d\theta(t_f)}{dt} = 0$

W-E: $g_{\dot{x}}$ and $g - g_{\dot{x}}\dot{x}$ are continuous

Good luck!