Exam 17.4.2023

In each task, you can get a maximum of 6 points for the correct solution. You can reach a maximum of 30 points in the exam.

1. Explain shortly.
   a. What are the costate variables?
   b. What is Pontryagin's maximum principle?
   c. What is the bang-bang control?
   d. What is a policy (DP)?
   e. What is the shortest path problem?
   f. What are reachable states?

2. Find the extremal of the functional
$$J = \int_0^1 [t\dot{x}(t) + x^2(t) + \dot{x}^2(t)]\, dt,$$
   using the calculus of variations, where $x(0) = 1$ and $x(1)$ is free.
   Explain the steps that you are taking.

3. Consider the state equation $\dot{x} = x + u$, $x(0) = 5$. Find the optimal control and the optimal state trajectory while maximizing the cost functional
$$J(u) = \int_0^2 (2x - 3u - u^2)\, dt,$$
   where $x(2)$ is free.

4. Let us examine the control problem
$$\dot{x}(t) = ax(t) + bu(t),\ x(0) = x_0 \in \mathbb{R}, \quad t \in [0, T],$$
   where $u \in \mathbb{R}$ is the control, T is known, and $a, b \in \mathbb{R}$ are parameters.
   Our aim is to minimize the cost
$$x^2(T) + \int_0^T [x^2(t) + u^2(t)]\, dt.$$
   Show that the cost-to-go which is of the following form
$$J(t, x) = k(t)x^2,$$
   where $k : [0, T] \to \mathbb{R}$ is a solution to the differential equation
$$\dot{k}(t) = -2ak(t) + b^2k^2(t) - 1, \quad k(T) = 1, \quad t \in [0, T],$$
   is an *optimal* cost-to-go.
   Find also the optimal control as a function of $k(t)$ and x.
   Hint: Use the HJB equation.

Please turn! ↪

5. Your task is to to find the optimal alignment for the two DNA-sequences:

   (1) G A A T T C A G T T A

   (2) G G A T C G T A

   To find out similarity of the sequences, we weight different alignments. $S_{i,j} = 1$ if base in sequence 1 at place j matches the base of sequence 2 at place i. We also assume that in evolution there are two types of mutations that happen more than other; C $\rightarrow$ T and T $\rightarrow$ C. Thus, also $S_{i,j} = 1$ if in sequence 1 in the place j is the structural unit T or C and in the sequence 2 in the place i is the structural unit C or T. In other cases, $S_{i,j} = 0$. There can also be gaps in sequence 1 or 2, which is penalized by $w = 0$.

   The task is to find the alignment with highest score. We design an alignment matrix whose elements are:

$$M_{i,j} = \max \left\{ \begin{array}{ll} M_{i-1,j-1} + S_{i,j}, & \text{(match/no match)} \\ M_{i,j-1} + w, & \text{(gap in sequence 2)} \\ M_{i-1,j} + w, & \text{(gap in sequence 1)} \end{array} \right\}$$

   The text in the bracket gives the "control" how the sequence (2) is built. Also assume that $i, j = 1, 2, ...$ and $M_{0,0} = M_{i,0} = M_{0,j} = 0$

   Using the DP-algorithm, we get the following scoring matrix

	G	A	A	T	T	C	A	G	T	T	A
G	1	1	1	1	1	1	1	1	1	1	1
G	1	1	1	1	1	1	1	2	2	2	2
A	1	2	2	2	2	2	2	2	2	2	3
T	1	2	2	3	3	3	3	3	3	3	3
C	1	2	2	3	4	4	4	4	4	4	4
G	1	2	2	3	4	4	4	5	5	5	5
T	1	2	2	3	4	5	5	5	6	6	6
A	1	2	3	3	4	5	6	6	6	6	7

   Your task is to find all optimal alignments (i.e., which alignments produce the score 6 in the scoring matrix) by backtracking from the scoring matrix.

Appendix

HJB: $0 = J_t + \min_{u(t)} \{g + J_x^T f\}$
E-L: $0 = g_x - \frac{d}{dt}(g_{\dot{x}})$
Hamiltonian: $H(x(t), u(t), t) = g(x(t), u(t), t) + p^T(t) f(x(t), u(t), t)$
costate: $\dot{p}(t) = -\frac{\partial H}{\partial x}(\cdot, t)$
free final state: $0 = g_{\dot{x}}(\cdot, t_f)$ or $\ell_x(x(t_f), t_f) - p(t_f) = 0$
free final time: $0 = g(\cdot, t_f) - g_{\dot{x}}\dot{x}(\cdot, t_f)$ or $H(\cdot, t_f) + \ell_t(x(t_f), t_f) = 0$
free final state and time: $g(\cdot, t_f) = g_{\dot{x}}(\cdot, t_f) = 0$ or $\ell_x(x(t_f), t_f) - p(t_f) = 0 = H(\cdot, t_f) + \ell_t(x(t_f), t_f)$
goal: $0 = g(\cdot, t_f) + g_{\dot{x}}(\cdot, t_f)^T \left[\frac{d\theta(t_f)}{dt} - \dot{x}(t_f)\right]$ or $H(\cdot, t_f) + \ell_t(x(t_f), t_f) + (\ell_x(x(t_f), t_f) - p(t_f))^T \frac{d\theta(t_f)}{dt} = 0$
W-E: $g_{\dot{x}}$ and $g - g_{\dot{x}}\dot{x}$ are continuous

Good luck!