Leskelä–Alaluusua

Examination

You may bring to the exam a handwritten *memory aid sheet* of size A4 with text only on one side, containing your name and student id in the upper right corner. You don't need to return your memory aid sheet. The exam contains 4 problems each worth 6 points.

1. Define a function $X : \Omega \to \mathbb{R}$ on $\Omega = \{1,2,3,4,5\}$ by setting $X(\omega) = 3 - |\omega - 3|$.

(a) Determine $\sigma(X)$. **(2 p)**

(b) Let $(\Omega, \sigma(\mathcal{D}), \mathbb{P})$ in which $\mathcal{D} = \{\{3\}, \{1,2,5\}, \{3,4,5\}\}$ and $\mathbb{P}(A) = \frac{\#A}{\#\Omega}$ for $A \in \sigma(\mathcal{D})$. Is X a random variable on probability space $(\Omega, \sigma(\mathcal{D}), \mathbb{P})$? If yes, explain why. If not, write down a Borel set $B \in \mathcal{B}(\mathbb{R})$ for which the preimage of X is not contained in $\sigma(\mathcal{D})$. Justify your answer in high detail. **(4 p)**

2. Let X and Y be independent and identically distributed random variables, both distributed according to the probability density function

$$f(x) = \begin{cases} e^{-x} & \text{for } x \geq 0, \\ 0 & \text{else,} \end{cases}$$

with respect to the Lebesgue measure on the real line. Define $S = X + \frac{1}{2}Y$ and $M = \max\{X, Y\}$.

(a) Determine the characteristic function $\phi_S(\theta) = \mathbb{E}e^{i\theta S}$. **(2 p)**

(b) Determine a probability density function $f_M(t) = \frac{d}{dt}\mathbb{P}(M \leq t)$ for M. **(1 p)**

(c) Determine the characteristic function $\phi_M(\theta) = \mathbb{E}e^{i\theta M}$. **(1 p)**

(d) Observe that $\phi_S = \phi_M$. What can you say about $\mathbb{P}(S = M)$? What can you conclude about the laws of S and M? **(2 p)**

Hint. $\int_{[0,\infty)} e^{-ax+ibx} \,\mathrm{Leb}(dx) = \frac{1}{a-ib}$ for all real numbers $a > 0$ and $b \in \mathbb{R}$ where Leb denotes the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

3. A measure μ on measurable space $(S, \mathcal{S})$ is called: (i) *σ-finite* if there exist measurable sets $S_1, S_2, \dots$ such that $\mu(S_n) < \infty$ for all n and $S_n \uparrow S$; (ii) *s-finite* if it can be written as a countable sum $\mu = \sum_{n=1}^{\infty} \mu_n$ where μ_n are finite measures.

(a) Assume that μ_n is a measure on $(S, \mathcal{S})$ for each n. Show that $\mu = \sum_{n=1}^{\infty} \mu_n$ defined by $\mu(B) = \sum_{n=1}^{\infty} \mu_n(B)$ is a measure on $(S, \mathcal{S})$. **(2 p)**

(b) Prove that any σ-finite measure is s-finite. **(2 p)**

(c) Write down an example of an s-finite measure which is not σ-finite. **(2 p)**

4. Let $U_1, U_2, \dots$ be mutually independent random variables, each one uniformly distributed on the interval $[0, 1]$. Define $M_n = \max(U_1, \dots, U_n)$. As $n \to \infty$, do the random variables M_n converge to a limit

(a) in distribution, **(2 p)**

(b) in probability, **(2 p)**

(c) almost surely? **(2 p)**

If your answer is yes, describe what the limit is in each case. If your answer is no, explain why a limit does not exist. In either case, justify your answer carefully in high detail.

Hint. Cumulative distribution functions, the lemmas of Émile Borel and Francesco Cantelli, or both, could be helpful.