



MS-A0402 Foundations of discrete mathematics solutions

Date of exam: 14.4.2022

Solved by: mathwiz.ai

Disclaimer:

These problems are solved by mathwiz.ai, a service leveraging the latest generative AI technologies. This means the solutions come with inherent limitations:

1. **Understanding:** The AI doesn't understand mathematics like humans do; it predicts answers based on historical data patterns.
2. **Reliability and Context:** The AI's responses can be contextually inappropriate or inconsistent, and thus contain errors.
3. **Creativity:** AI applies known patterns, it does not invent new mathematical methods.

Use these solutions as guides, not definitive answers. Despite occasional errors, the solutions can be useful for providing context around the problems. For verifiable accurate solutions, seek guidance from a qualified educator. At the time of solving these problems, the mathematical capabilities of AIs were at the level of an advanced university mathematics student. If you find errors in the solutions, feel free to inform us. Contact information: mathwizcontact@gmail.com.

Problem 1

a)

Define the cartesian product $S \times T$ of sets S and T .

b)

Define the Power set $P(A)$ of a set A .

c)

Let $S = \{a, b\}$ and $T = \{1, 2\}$. Write down

$$P(S \times T)$$

c)

Prove that for all non-empty sets A, B, C we have

$$A \times (B \cup C) \subset (A \times B) \cup (A \times C)$$

Problem 1 solution

a) The Cartesian product of two sets S and T , denoted by $S \times T$, is the set of all ordered pairs (s, t) , where $s \in S$ and $t \in T$. Formally,

$$S \times T = \{(s, t) \mid s \in S, t \in T\}$$

b) The power set of a set A , denoted by $P(A)$, is the set of all subsets of A , including the empty set and the set A itself. Formally,

$$P(A) = \{B \mid B \subseteq A\}$$

c) Let $S = \{a, b\}$ and $T = \{1, 2\}$. We first find the Cartesian product $S \times T$:

$$S \times T = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$$

Now, we can find the power set $P(S \times T)$:

$$P(S \times T) = \{\emptyset, \{(a, 1)\}, \{(a, 2)\}, \{(b, 1)\}, \{(b, 2)\}, \{(a, 1), (a, 2)\}, \{(a, 1),$$

$$(b, 1)\}, \{(a, 1), (b, 2)\}, \{(a, 2), (b, 1)\}, \{(a, 2), (b, 2)\}, \{(b, 1), (b, 2)\}, \{(a, 1), (a, 2),$$

$(b, 1)\}, \{(a, 1), (a, 2), (b, 2)\}, \{(a, 1), (b, 1), (b, 2)\}, \{(a, 2), (b, 1), (b, 2)\}, \{(a, 1), (a, 2), (b, 1), (b, 2)\}\}$

d) We need to prove that for all non-empty sets A, B, C , we have

$$A \times (B \cup C) \subset (A \times B) \cup (A \times C)$$

Let $(a, x) \in A \times (B \cup C)$. This means that $a \in A$ and $x \in B \cup C$. Since $x \in B \cup C$, we have either $x \in B$ or $x \in C$.

If $x \in B$, then $(a, x) \in A \times B$, and thus $(a, x) \in (A \times B) \cup (A \times C)$.

If $x \in C$, then $(a, x) \in A \times C$, and thus $(a, x) \in (A \times B) \cup (A \times C)$.

In both cases, we have $(a, x) \in (A \times B) \cup (A \times C)$. Therefore, $A \times (B \cup C) \subset (A \times B) \cup (A \times C)$.

Problem 2

a)

Prove that for all odd integers n , then n^2 is also odd.

b)

Prove, using induction, that for all $n = 1, 2, 3, \dots$,

$$\sum_{i=1}^n (2i - 1) = n^2$$

Problem 2 solution

a) To prove that for all odd integers n , n^2 is also odd, let's first recall the definition of odd integers. An integer n is odd if there exists an integer k such that $n = 2k + 1$.

Now, let n be an odd integer. Then, there exists an integer k such that $n = 2k + 1$. We want to show that n^2 is also odd. Let's compute n^2 :

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

Let $m = 2k^2 + 2k$. Then, $n^2 = 2m + 1$, where m is an integer. This means that n^2 is also odd. Therefore, for all odd integers n , n^2 is also odd.

b) We want to prove, using induction, that for all $n = 1, 2, 3, \dots$,

$$\sum_{i=1}^n (2i - 1) = n^2$$

Base case ($n = 1$):

$$\sum_{i=1}^1 (2i - 1) = 2(1) - 1 = 1 = 1^2$$

The base case holds.

Inductive step: Assume that the statement is true for some $n = k$, i.e.,

$$\sum_{i=1}^k (2i - 1) = k^2$$

We want to show that the statement is also true for $n = k + 1$:

$$\sum_{i=1}^{k+1} (2i - 1) = (k + 1)^2$$

Let's compute the sum for $n = k + 1$:

$$\begin{aligned} \sum_{i=1}^{k+1} (2i - 1) &= \sum_{i=1}^k (2i - 1) + (2(k + 1) - 1) \\ &= k^2 + 2(k + 1) - 1 \quad (\text{by the inductive hypothesis}) \\ &= k^2 + 2k + 2 - 1 \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2 \end{aligned}$$

Thus, the statement is true for $n = k + 1$.

By the principle of mathematical induction, the statement is true for all $n = 1, 2, 3, \dots$

Problem 3

a) Write the statement of the Binomial theorem.

b)

Write down the first 6 rows of the Pascal's triangle.

c)

What is the coefficient of x^2y^3 in the expansion of $(x + y)^5$?

Problem 3 solution

a) The Binomial theorem states that for any non-negative integer n and any real numbers a and b , the expansion of $(a + b)^n$ is given by:

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

where $\binom{n}{k}$ is the binomial coefficient, which is defined as:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

b) The first 6 rows of Pascal's triangle are:

```

“ 1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
“

```

c) To find the coefficient of x^2y^3 in the expansion of $(x + y)^5$, we can use the Binomial theorem. We need to find the term in the expansion where $n - k = 2$ and $k = 3$. From the Binomial theorem, the term is:

$$\binom{5}{3} x^2 y^3 = \frac{5!}{3!2!} x^2 y^3 = 10x^2 y^3$$

So, the coefficient of x^2y^3 in the expansion of $(x + y)^5$ is 10.

Problem 4

- a) Write the permutation $(1362)(2564)(2345)$
- as a product of disjoint cycles.
 - in two-line notation.
 - as a product of transpositions.
- b) (7pts)
- There are 6 participants who are first paired up for a dance. Afterwards, they pair up to play a game, where for some reason it is important that the two people in each pair did not dance with each other. In how many ways can this be done?
 - Solve the same problem when there were initially $2n$ participants for $n = 1, 2, 3, \dots$

Problem 4 solution

a) Let's find the product of the given cycles:

i) As a product of disjoint cycles:

$$(1362)(2564)(2345)$$

First, we multiply the last two cycles:

$$(2564)(2345) = (25)(364)$$

Now, we multiply the result with the first cycle:

$$(1362)(25)(364) = (13)(62)(25)(364)$$

So, the permutation as a product of disjoint cycles is $(13)(62)(25)(364)$.

ii) In two-line notation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 1 & 5 & 2 & 4 \end{pmatrix}$$

iii) As a product of transpositions:

$$(13)(62)(25)(364) = (13)(62)(25)(36)(34)$$

b)

i) For 6 participants, we can use the concept of derangements to find the number of ways they can be paired up for the game such that no one is paired with their dance partner. A derangement is a permutation of a set of elements such that no element appears in its original position. The number of derangements of a set of size n is denoted by D_n .

For $n = 3$ (6 participants), we have:

$$D_3 = 3!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}) = 3!(1 - 1 + \frac{1}{2} - \frac{1}{6}) = 3!(\frac{1}{2} - \frac{1}{6}) = 3!(\frac{2}{6}) = 6 \cdot \frac{2}{6} = 2$$

So, there are 2 ways to pair up the 6 participants for the game such that no one is paired with their dance partner.

ii) For $2n$ participants, we can use the same concept of derangements to find the number of ways they can be paired up for the game such that no one is paired with their dance partner. The number of derangements of a set of size n is denoted by D_n .

For n ($2n$ participants), we have:

$$D_n = n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!})$$

The number of ways to pair up the $2n$ participants for the game such that no one is paired with their dance partner is given by the derangement formula D_n .

Problem 5

a)

Compute (with proof) the chromatic number of the Grötzsch graph, depicted below.

b)

Find all the solutions $x, y \in \mathbb{Z}$ for the Diophantine equation

$$514x + 387y = 2.$$

Problem 5 solution

a) The Grötzsch graph is a triangle-free graph with 11 vertices and 20 edges. It is a well-known example of a graph with chromatic number 4. To prove this, we will show that the Grötzsch graph cannot be colored with 3 colors, and then provide a 4-coloring.

Suppose, for the sake of contradiction, that the Grötzsch graph can be colored with 3 colors. Then, each of its five 4-cycles must have at least one pair of adjacent vertices with the same color. Let's label the vertices of the Grötzsch graph as follows:

```

    “ a—b
  / / c—d—e
  / / f—g
    “

```

Without loss of generality, assume that vertices a and b have the same color. Then, vertices c and d must have different colors, say c has color 2 and d has color 3. Now, vertex e cannot have color 2 or 3, because it is adjacent to both c and d. But it also cannot have the same color as a and b, because it is adjacent to b. This is a contradiction, so the Grötzsch graph cannot be colored with 3 colors.

Now, we provide a 4-coloring for the Grötzsch graph:

- Color 1: a, d, g
- Color 2: b, c, f
- Color 3: e
- Color 4: (the center vertex)

This coloring uses 4 colors and satisfies the condition that adjacent vertices have different colors. Therefore, the chromatic number of the Grötzsch graph is 4.

b)

We need to find all the solutions $x, y \in \mathbb{Z}$ for the Diophantine equation $514x + 387y = 2$.

First, we find the greatest common divisor (GCD) of 514 and 387 using the Euclidean algorithm:

$$\begin{aligned} 514 &= 1 \cdot 387 + 127 \\ 387 &= 3 \cdot 127 + 6 \\ 127 &= 21 \cdot 6 + 1 \end{aligned}$$

The GCD of 514 and 387 is 1. Since 1 divides 2, there exist integer solutions for the given Diophantine equation.

Now, we can use the extended Euclidean algorithm to find a particular solution:

$$\begin{aligned} 1 &= 127 - 21 \cdot 6 \\ &= 127 - 21 \cdot (387 - 3 \cdot 127) \\ &= 64 \cdot 127 - 21 \cdot 387 \\ &= 64 \cdot (514 - 1 \cdot 387) - 21 \cdot 387 \\ &= 64 \cdot 514 - 85 \cdot 387 \end{aligned}$$

We already found a particular solution for the equation $514x + 387y = 1$, which is $x_0 = 64$ and $y_0 = -85$. Now, we can multiply both sides of the equation by 2 to get the equation $514(2x) + 387(2y) = 2$, or $1028x + 774y = 2$.

The particular solution for this equation is $x_1 = 2 \cdot 64 = 128$ and $y_1 = 2 \cdot (-85) = -170$.

Now, we can use the general solution for Diophantine equations of the form $ax + by = c$, which is given by:

$$x = x_1 + \frac{b}{\gcd(a, b)}t, \quad y = y_1 - \frac{a}{\gcd(a, b)}t$$

In this case, $a = 1028$, $b = 774$, $x_1 = 128$, $y_1 = -170$, and $\gcd(a, b) = 2$. So, the general solution is:

$$x = 128 + 387t, \quad y = -170 - 514t$$

where $t \in \mathbb{Z}$. This gives us all the solutions for the Diophantine equation $514x + 387y = 2$.