ms-a0503 First course in probability and statistics model solutions

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Problem 1

Ville and Kalle ride separate buses to the city center in order to meet. Their buses start at the same time. Ville's travel time has normal distribution with mean 20 minutes and standard deviation 3 minutes. Kalle's travel time has normal distribution with mean 22 minutes and standard deviation 3 minutes. The travel times are independent.
Let X denote how much Kalle arrives after Ville (negative if Kalle arrives first).
(a)
What is the distribution of X ? (Give the name and parameters.) Why?
(b)
What is the probability that Kalle arrives first?
(c)
What is the probability that one of them has to wait more than 3 minutes?

Problem 1 solution

(a)
What is the distribution of X? (Give the name and parameters.) Why?

Since both Ville's and Kalle's travel times are normally distributed and independent, the difference in their travel times, X, is also normally distributed. The mean of X is the difference in the means of their travel times, and the variance of X is the sum of their variances.

Mean of X: $\mu_X = \mu_{Kalle} - \mu_{Ville} = 22 - 20 = 2$

Variance of X: $\sigma_X^2 = \sigma_{Kalle}^2 + \sigma_{Ville}^2 = 3^2 + 3^2 = 18$

Standard deviation of X: $\sigma_X = \sqrt{18} \approx 4.24$

So, the distribution of X is a normal distribution with mean 2 and standard deviation 4.24, denoted as $X \sim N(2, 4.24^2)$.

(b) What is the probability that Kalle arrives first?

To find the probability that Kalle arrives first, we need to find the probability that $X < 0$. Since X is normally distributed, we can standardize it and use the standard normal distribution table (or a calculator) to find the probability:

$$P(X < 0) = P\left(\frac{X - \mu_X}{\sigma_X} < \frac{0 - 2}{4.24}\right) = P(Z < -0.472)$$

Using the standard normal distribution table or a calculator, we find that

$P(Z < -0.472) \approx 0.319$.

So, the probability that Kalle arrives first is approximately 0.319.

(c) What is the probability that one of them has to wait more than 3 minutes?

We want to find the probability that $|X| > 3$, which is the same as finding the probability that $X < -3$ or $X > 3$. We can use the standard normal distribution again:

$$P(X < -3) = P\left(\frac{X - \mu_X}{\sigma_X} < \frac{-3 - 2}{4.24}\right) = P(Z < -1.18)$$

$$P(X > 3) = P\left(\frac{X - \mu_X}{\sigma_X} > \frac{3 - 2}{4.24}\right) = P(Z > 0.236)$$

Using the standard normal distribution table or a calculator, we find that $P(Z < -1.18) \approx 0.119$ and $P(Z > 0.236) \approx 0.407$.

The probability that one of them has to wait more than 3 minutes is the sum of these probabilities:

$$P(|X| > 3) = P(X < -3) + P(X > 3) \approx 0.119 + 0.407 = 0.526$$

So, the probability that one of them has to wait more than 3 minutes is approximately 0.526.

Problem 2

The engine of a spaceship runs for a time T (seconds) and it causes the acceleration $a = 2$ m/s^2. The distance travelled during acceleration is $X = \frac{1}{2}aT^2$. The time T is a continuous random variable uniformly distributed in the interval $I = [100, 150 + r]$ (seconds), where r is the first digit of your student number (one of $0, \ldots, 9$, ignore possible letters). Write the value of r clearly in your answer and use this value in your calculations.

(a) Calculate the expected value of the distance travelled.
(b) Calculate the probability that the distance is more than 20000 meters.
(c) Calculate the standard deviation of the distance.
(d) Ten similar spaceships are sent, and their engines are run for times $T_1, \ldots, T_{10}$, which are independent and uniformly distributed over the same interval I. Calculate the probability that exactly three of the ships travel more than 20000 meters during the acceleration. (1p)

Problem 2 solution

Let r be a given value. In this case, the interval I is $[100, 150 + r]$.

(a) Calculate the expected value of the distance traveled.

The expected value of the time T is:

$$E[T] = \frac{1}{2}(100 + 150 + r) = \frac{250 + r}{2}$$

The expected value of the distance traveled X is:

$$E[X] = E\left[\frac{1}{2}aT^2\right] = \frac{1}{2}aE[T^2]$$

To find $E[T^2]$, we can use the formula for the second moment of a uniform distribution:

$$E[T^2] = \frac{1}{3}(100^2 + 100(150 + r) + (150 + r)^2)$$

Now, we can find the expected value of the distance traveled:

$$E[X] = \frac{1}{2}(2)\left(\frac{1}{3}(100^2 + 100(150 + r) + (150 + r)^2)\right) = \frac{1}{3}(100^2+100(150+r)+(150+r)^2)$$

(b) Calculate the probability that the distance is more than 20000 meters.

We want to find the probability that $X > 20000$. First, we need to find the corresponding value of T:

$$20000 = \frac{1}{2}(2)T^2 \Rightarrow T^2 = 20000 \Rightarrow T = \sqrt{20000} \approx 141.42$$

Now, we can find the probability that the time is greater than this value:

$$P(T > 141.42) = \frac{(150 + r) - 141.42}{(150 + r) - 100}$$

(c) Calculate the standard deviation of the distance.

First, we need to find the variance of the time T:

$$Var(T) = E[T^2] - (E[T])^2 = \frac{1}{3}(100^2 + 100(150 + r) + (150 + r)^2) - \left(\frac{250 + r}{2}\right)^2$$

Now, we can find the variance of the distance traveled X:

$$Var(X) = \left(\frac{1}{2}a\right)^2 Var(T) = (1)\left(\frac{1}{3}(100^2 + 100(150 + r) + (150 + r)^2) - \left(\frac{250 + r}{2}\right)^2\right)$$

The standard deviation of the distance is the square root of the variance:

$$SD(X) = \sqrt{Var(X)}$$

(d) Ten similar spaceships are sent, and their engines are run for times $T_1, \ldots, T_{10}$, which are independent and uniformly distributed over the same interval I. Calculate the probability that exactly three of the ships travel more than 20000 meters during the acceleration.

Let p be the probability that a single ship travels more than 20000 meters, which we found in part (b). Since the ships are independent, we can model this situation as a binomial distribution with $n = 10$ trials and probability of success p. We want to find the probability that exactly 3 ships travel more than 20000 meters:

$$P(X = 3) = \binom{10}{3} p^3 (1 - p)^7$$

Problem 3

60% of Finns are "young" (less than 50 years) and the rest are "old". 35% of the young and 85% of the old use eyeglasses. From the population we take a random sample of 100 persons with replacement. The random variable X_i is 1 if the i th person in the sample is old, otherwise 0 . The random variable Y_i is 1 if the i th person uses eyeglasses, otherwise 0 . Furthermore X is the number of old people in the sample and Y is the number of people using eyeglasses in the sample.

(a) Calculate $\mathbb{E}\left(X_1\right)$ and $\mathbb{E}(X)$.
(b) Calculate $\mathbb{E}\left(Y_1\right)$ and $\mathbb{E}(Y)$.
(c) Calculate $\operatorname{Cov}\left(X_1, Y_1\right)$.
(d) Calculate $\operatorname{Cov}\left(X_1, Y_2\right)$.
(e) Calculate $\operatorname{Cov}(X, Y)$.

Problem 3 solution

(a) Calculate $\mathbb{E}(X_1)$ and $\mathbb{E}(X)$.

X_1 is 1 if the 1st person in the sample is old, otherwise 0. The probability that a person is old is 40

$$\mathbb{E}(X_1) = 1 \cdot P(X_1 = 1) + 0 \cdot P(X_1 = 0) = 0.4$$

Since the sample has 100 persons, the expected number of old people in the sample is:

$$\mathbb{E}(X) = 100 \cdot \mathbb{E}(X_1) = 100 \cdot 0.4 = 40$$

(b) Calculate $\mathbb{E}(Y_1)$ and $\mathbb{E}(Y)$.

Y_1 is 1 if the 1st person in the sample uses eyeglasses, otherwise 0. The probability that a young person uses eyeglasses is 35%, and the probability that an old person uses eyeglasses is 85%. Since 60% of the population is young and 40% is old, the probability that a person uses eyeglasses is:

$$P(Y_1 = 1) = 0.6 \cdot 0.35 + 0.4 \cdot 0.85 = 0.21 + 0.34 = 0.55$$

So, the expected value of Y_1 is:

$$\mathbb{E}(Y_1) = 1 \cdot P(Y_1 = 1) + 0 \cdot P(Y_1 = 0) = 0.55$$

Since the sample has 100 persons, the expected number of people using eyeglasses in the sample is:

$$\mathbb{E}(Y) = 100 \cdot \mathbb{E}(Y_1) = 100 \cdot 0.55 = 55$$

(c) Calculate $\mathrm{Cov}(X_1, Y_1)$.
Covariance is defined as:

$$\mathrm{Cov}(X_1, Y_1) = \mathbb{E}(X_1 Y_1) - \mathbb{E}(X_1)\mathbb{E}(Y_1)$$

We need to find $\mathbb{E}(X_1 Y_1)$, which is the expected value of the product of X_1 and Y_1. Since $X_1 = 1$ when the person is old and $Y_1 = 1$ when the person uses eyeglasses, $X_1 Y_1 = 1$ when the person is old and uses eyeglasses. The probability of this happening is:

$$P(X_1 Y_1 = 1) = P(\text{old and uses eyeglasses}) = P(\text{old}) \cdot P(\text{uses eyeglasses}|\text{old}) = 0.4 \cdot 0.85 = 0.34$$

So, $\mathbb{E}(X_1 Y_1) = 0.34$. Now we can find the covariance:

$$\mathrm{Cov}(X_1, Y_1) = \mathbb{E}(X_1 Y_1) - \mathbb{E}(X_1)\mathbb{E}(Y_1) = 0.34 - 0.4 \cdot 0.55 = 0.34 - 0.22 = 0.12$$

(d) Calculate $\mathrm{Cov}(X_1, Y_2)$.
Since the persons in the sample are independent, the covariance between X_1 and Y_2 is 0:

$$\mathrm{Cov}(X_1, Y_2) = 0$$

(e) Calculate $\mathrm{Cov}(X, Y)$.
The covariance between X and Y can be found using the formula:

$$\mathrm{Cov}(X, Y) = \sum_{i=1}^{100} \sum_{j=1}^{100} \mathrm{Cov}(X_i, Y_j)$$

Since the covariance between different persons is 0, we only need to consider the covariance between the same person:

$$\mathrm{Cov}(X, Y) = \sum_{i=1}^{100} \mathrm{Cov}(X_i, Y_i) = 100 \cdot \mathrm{Cov}(X_1, Y_1) = 100 \cdot 0.12 = 12$$

Problem 4

An exam in Skepticism contains six questions. In each question the student has to choose one of two possible answers. Each correct answer yields one point. 2/3 of the participants are hard-working and the remaining ones are lazy. A hard-working student answers each question correctly with probability 0.9 , independent of the other questions. A lazy student chooses each answer uniformly at random from the two choices and gets it right with probability 0.5 .

(a)

If a student is hard-working, what is the probability that he/she gets exactly 3 points?

(b)

If a student is lazy, what is the probability that he/she gets exactly 3 points?

(c)

If a student gets exactly 3 points, what is the probability that he/she is lazy?

Problem 4 solution

(a) If a student is hard-working, what is the probability that he/she gets exactly 3 points?

A hard-working student answers each question correctly with probability 0.9. Therefore, the probability of answering a question incorrectly is 0.1. Since there are 6 questions, we can model this situation as a binomial distribution with $n = 6$ trials and probability of success (answering correctly) $p = 0.9$. We want to find the probability that the student gets exactly 3 points:

$$P(X = 3|\text{hard-working}) = \binom{6}{3}(0.9)^3(0.1)^3 = 20 \cdot 0.729 \cdot 0.001 = 0.01458$$

So, if a student is hard-working, the probability that he/she gets exactly 3 points is approximately 0.01458.

(b) If a student is lazy, what is the probability that he/she gets exactly 3 points?

$$P(X = 3|\text{lazy}) = \binom{6}{3}(0.5)^3(0.5)^3 = \frac{20}{64} \approx 0.3125$$

(c) If a student gets exactly 3 points, what is the probability that he/she is lazy?

We can use Bayes' theorem to find the probability that a student is lazy given that he/she gets exactly 3 points:

$$P(\text{lazy}|X = 3) = \frac{P(X = 3|\text{lazy})P(\text{lazy})}{P(X = 3)}$$

We need to find the probability $P(X = 3)$, which can be calculated as:

$$P(X = 3) = P(X = 3|\text{hard-working})P(\text{hard-working}) + P(X = 3|\text{lazy})P(\text{lazy}) = 0.01458 \cdot \frac{2}{3} + 0.3125 \cdot \frac{1}{3} \approx 0.1042$$

Now we can find the probability that a student is lazy given that he/she gets exactly 3 points:

$$P(\text{lazy}|X = 3) = \frac{P(X = 3|\text{lazy})P(\text{lazy})}{P(X = 3)} = \frac{0.3125 \cdot \frac{1}{3}}{0.1042} \approx 0.998$$

So, if a student gets exactly 3 points, the probability that he/she is lazy is approximately 0.998.