Instructions: Answer as many questions as possible. Each subquestion (labelled with letters) carries equal weight, and is worth a maximum mark of 6 points.

It is only permitted to bring to the exam room basic writing material and a scientific calculator.

1. Let $A = U\Sigma V^T \in \mathbb{R}^{3\times 2}$ where

$$U = \begin{bmatrix} 3/5 & 4/5 & 0 \\ 4/5 & -3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, V = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Moreover, let $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \in \mathbb{R}^3$.

(a) Compute A. Then, explain why $A = U\Sigma V^T$ is a singular value decomposition of A. Check explicitly that the three matrices U, Σ and V have the required properties.

(b) Using part (a) or otherwise, compute the Moore-Penrose pseudoinverse of the matrix A. Then, compute the minimum norm solution to the least square problem $\min_{\mathbf{x}\in\mathbb{R}^2} \|A\mathbf{x} - \mathbf{b}\|_2$.

(c) Using parts (a) and (b) or otherwise, compute a basis for $N(A)$, the null space of the matrix A. Then, describe the set of *all* solutions to the least square problem $\min_{\mathbf{x}\in\mathbb{R}^2} \|A\mathbf{x} - \mathbf{b}\|_2$.

2. Let

$$B = \begin{bmatrix} y & y^2 & 1 \\ 0 & y & 0 \\ y & -y & 1 \end{bmatrix}$$

where $y \in \mathbb{R}$ is a parameter.

(a) Give the definition of $R(M)$, the range of a matrix M, and the definition of $N(M)$, the null space of a matrix M. Then, find bases for both $R(B)$ and $N(B)$ in the special case $y = 0$ and in the special case $y = 1$.

(b) Consider the right hand side $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. For which values of y does the linear system $B\mathbf{x} = \mathbf{b}$ admit a solution (not necessarily unique)? Justify your answer. Then, compute all solutions to $B\mathbf{x} = \mathbf{b}$ for the special values of y for which they exist.

3. Two square matrices $A, B \in \mathbb{C}^{n \times n}$ are said to be *simultaneously diagonalisable* if there exist an invertible matrix $X \in \mathbb{C}^{n \times n}$ and two diagonal matrices $\Lambda_A, \Lambda_B \in \mathbb{C}^{n \times n}$ such that $A = X\Lambda_A X^{-1}$ and $B = X\Lambda_B X^{-1}$. In other words, two simultaneously diagonalisable matrices are both diagonalisable with the *same* similarity matrix X.

   (a) Let $A, B \in \mathbb{C}^{n \times n}$ be two square matrices.

      (i) Prove that, if A and B are simultaneously diagonalisable, then $AB = BA$;

      (ii) Show, by providing a counterexample, that it can happen that $AB = BA$ but A and B are not simultaneously diagonalisable. *Hint: If A is not diagonalisable, then clearly A and B cannot be simultaneously diagonalisable.*

   (b) Prove that, for every $m, n \in \mathbb{N}$ and every $a, b \in \mathbb{C}$, the matrices

$$A = \begin{bmatrix} a^m & b^m - a^m \\ 0 & b^m \end{bmatrix}, \qquad B = \begin{bmatrix} a^n & b^n - a^n \\ 0 & b^n \end{bmatrix} \in \mathbb{C}^{2\times 2}.$$

   are simultaneously diagonalisable. *Hint: Compute eigenvalues and eigenvectors of both A and B. You may analyse separately the cases $a = b$ and $a \neq b$.*