Mathematics
Aalto University

Hannukainen/Autio

MS-E1651 - Numerical Matrix Computations

Exam 18.10.2023

Please fill in clearly *on every sheet* the data on you and the examination. On *Examination code* mark course code, title and text mid-term or final examination.

The exam time is three hours (unless otherwise agreed before the exam). No electronic calculators or materials are allowed.

Solve all problems. The grade is best of the two:

- Grade is based only on the exam.
- Grade is based on exercise points and exam points.

1. (a) (3p) Show that the product of two $n \times n$ lower triangular matrices is a lower triangular matrix using an induction proof with respect to dimension.

   (b) (3p) Give the definition of a positive definite matrix. Let $E \in \mathbb{R}^{n\times n}$ be s.t. $E = E^T$ and $\|E\|_2 = 2$. For which $\alpha \in \mathbb{R}$ can you guarantee that

$$\alpha I + E$$

   is symmetric and positive definite? Hint: use the properties of the operator norm and the inequality $\boldsymbol{v}^T E\boldsymbol{w} \geq -\|\boldsymbol{v}\|_2 \|E\boldsymbol{w}\|_2$ for any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$.

2. Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{p}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \boldsymbol{p}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{p}_3 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}.$$

   (a) (2p) Verify that $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ are A-orthogonal.

   (b) (2p) Show that the A-orthogonal projection onto $span(\boldsymbol{p}_1, \boldsymbol{p}_2)$ has the expression

$$P_A = \frac{\boldsymbol{p}_1 \boldsymbol{p}_1^T A}{\|\boldsymbol{p}_1\|_A^2} + \frac{\boldsymbol{p}_2 \boldsymbol{p}_2^T A}{\|\boldsymbol{p}_2\|_A^2}$$

   (c) (2p) Let $\boldsymbol{x} = \boldsymbol{p}_1 + \boldsymbol{p}_2 + \pi\sqrt{99}\boldsymbol{p}_3$. Find $\hat{\boldsymbol{x}} \in span(\boldsymbol{p}_1, \boldsymbol{p}_2)$ s.t.

$$\|\boldsymbol{x} - \hat{\boldsymbol{x}}\|_A < \|\boldsymbol{x} - \hat{\boldsymbol{x}} + \boldsymbol{v}\|_A \quad \text{for any } \boldsymbol{v} \in span(\boldsymbol{p}_1, \boldsymbol{p}_2),\ \boldsymbol{v} \neq 0.$$

3. (a) Let $A_1 \in \mathbb{R}^{n_1 \times n_1}$, $A_2 \in \mathbb{R}^{n_2 \times n_2}$ be s.p.d. matrices with Cholesky factors L_1, L_2. Give the Cholesky factorisation of

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

(b) Compute the Cholesky factor of

$$\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

(c) Give the sparse Cholesky factorisation of

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 5 \end{bmatrix}$$

Use the permutation matrix P corresponding to the permutation vector $\begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$.

4. (a) (3p) Let $a, b, c \in \mathbb{R}$. Show that

$$-\left|\frac{ab}{c}\right|(2u + u^2) \leq fl\left(\frac{ab}{c}\right) - \frac{ab}{c} \leq \left|\frac{ab}{c}\right|(2u + u^2) \quad (2p)$$

where u is the machine epsilon.

(b) (3p) Let $A_1, A_2, A_3 \in \mathbb{R}^{10\times 10}$ be defined as

$$A_1 = QDQ^T, \quad A_2 = A_1^2, \quad \text{and} \quad A_3 = dI + A_1$$

Here $Q \in \mathbb{R}^{10\times 10}$ is a unitary matrix, $D \in \mathbb{R}^{10\times 10}$ is a diagonal matrix with entries $D_{ii} = i$, and $d = 10^6$. Consider solving systems $A_i\boldsymbol{x} = \boldsymbol{b}$ for $i \in \{1, 2, 3\}$ in floating-point representation. Order the systems based on how accurately you expect that they can be solved. Justify your ordering.