# ELEC-A7200 Signals and Systems, fall 2023

## Midterm exam/välikoe/mellanprov I

### Problem/Tehtävä/Problem 1

Figure 1a and 1b illustrates the two-sided amplitude and phase spectrum, respectively, of a periodic signal $x_1(t)$. Let the signal periodicity $T_0 = 1$ ms.

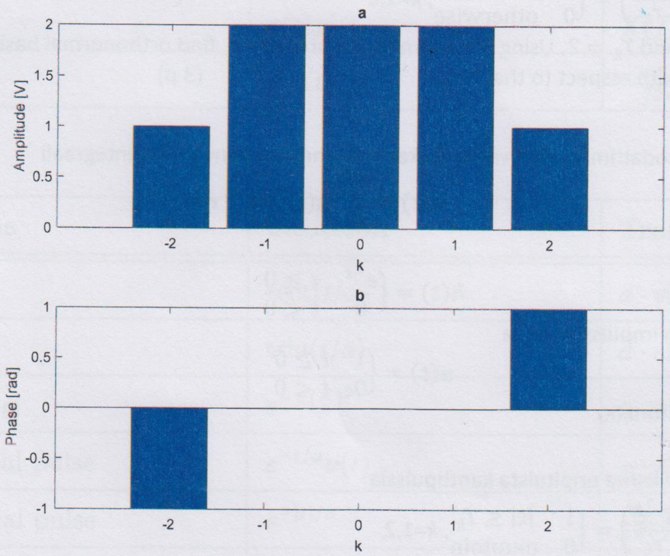
a) Determine the mean power of the signal. (2 p)
b) Express $x_1(t)$ using trigonometric functions. (2 p)
c) Sketch the two-sided power spectrum of the signal. (1 p)
d) Sketch the one-sided power spectrum of the signal. (1 p)

Erään jaksollisen signaalin $x_1(t)$ amplitudi- ja vaihespektrit on esitetty kuvissa 1a ja 1b. Olkoon signaalin jaksonaika $T_0 = 1$ ms.

a) Ratkaise signaalin keskimääräinen teho. (2 p)
b) Esitä signaali $x_1(t)$ käyttäen trigonometrisiä funktioita. (2 p)
c) Hahmottele signaalin kaksipuolinen tehospektri. (1 p)
d) Hahmottele signaalin yksipuolinen tehospektri. (1 p)

Figur 1a och 1b illustrerar den tvåsidiga amplitud- och fasspektrumet, respektive, för en periodisk signal $x_1(t)$. Låt signalens periodicitet $T_0 = 1$ ms.

a) Beräkna signalens medeleffekt. (2 p)
b) Presentera signalen $x_1(t)$ med hjälp av trigonometriska funktioner. (2 p)
c) Skissa signalens tvåsidiga effektspektrum. (1 p)
d) Skissa signalens ensidiga effektspektrum. (1 p)




Figure/Kuva/Figur 1

### Problem/Tehtävä/Problem 2

Consider a triangular pulse

$$x_2(t) = \begin{cases} 1 - \frac{2}{T_0}|t| & |t| \le \frac{T_0}{2} \\ 0 & |t| > \frac{T_0}{2} \end{cases}$$

a) Determine the time derivative $x_3(t) = \frac{d}{dt}x_2(t)$ of the pulse $x_2(t)$. (2 p)
b) Determine the Fourier transform of the pulse $x_3(t)$. (2 p)
c) Determine the Fourier transform of the pulse $x_2(t)$. (2 p)

Tarkastellaan kolmiopulssia

$$x_2(t) = \begin{cases} 1 - \frac{2}{T_0}|t| & |t| \le \frac{T_0}{2} \\ 0 & |t| > \frac{T_0}{2} \end{cases}$$

a) Ratkaise pulssin $x_2(t)$ aikaderivaatta $x_3(t) = \frac{d}{dt}x_2(t)$. (2 p)
b) Ratkaise pulssin $x_3(t)$ Fourier-muunnos. (2 p)
c) Ratkaise pulssin $x_2(t)$ Fourier-muunnos (2 p)

Betrakta en trianglepuls

$$x_2(t) = \begin{cases} 1 - \frac{2}{T_0}|t| & |t| \le \frac{T_0}{2} \\ 0 & |t| > \frac{T_0}{2} \end{cases}$$

a) Bestäm tidsderivatan $x_3(t) = \frac{d}{dt}x_2(t)$ av pulsen $x_2(t)$. (2 p)

b) Bestäm Fouriertransformen av pulsen $x_3(t)$ (2 p)

c) Bestäm Fouriertransformen av pulsen $x_2(t)$ (2 p)

## Problem/Tehtävä/Problem 3

a) Determine the step response of a RC filter by solving the following convolution integral

$$s(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$$

Where

$$h(t) = \begin{cases} e^{-t} & t \ge 0 \\ 0 & t < 0 \end{cases}$$

denotes the impulse response of the filter and

$$u(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$$

Denotes the unit step function. (3 p)

b) Consider two rectangular pulses of different length

$$p_k(t) = \text{rect}\left(\frac{t-\frac{T_k}{2}}{T_k}\right) = \begin{cases} 1 & |t| \le T_k \\ 0 & \text{otherwise} \end{cases}, k=1,2$$

where $T_1 = 1$ and $T_2 = 2$. Using the Gram-Smith procedure, find orthonormal basis for the pulses and express them with respect to that basis. (3 p)

a) Ratkaise RC -suodattimen askelvaste, laskemalla oheinen konvoluutiointegraali

$$s(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$$

missä

$$h(t) = \begin{cases} e^{-t} & t \ge 0 \\ 0 & t < 0 \end{cases}$$

on suodattimen impulssivaste ja

$$u(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$$

on yksikköaskelfunktio. (3 p)

b) Tarkastellaan kahta eripituista kanttipulssia

$$p_k(t) = \text{rect}\left(\frac{t-\frac{T_k}{2}}{T_k}\right) = \begin{cases} 1 & |t| \le T_k \\ 0 & \text{muutoin} \end{cases}, k=1,2$$

jossa $T_1 = 1$ ja $T_2 = 2$. Ratkaise pulseille ortonormaalit kantafunktiot käyttäen Gram-Smith proseduuria ja esitä signaali näiden kantafunktioiden avulla (3 p)

a) Bestäm stegsvaret för ett RC-filter genom att lösa följande faltningsintegral

$$s(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$$

där

$$h(t) = \begin{cases} e^{-t} & t \ge 0 \\ 0 & t < 0 \end{cases}$$

betecknar filtrets impulssvar och

$$u(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$$

betecknar enhetsstegsfunktionen. (3 p)

b) Betrakta två rektangulära pulser av olika längd

$$p_k(t) = \text{rect}\left(\frac{t-\frac{T_k}{2}}{T_k}\right) = \begin{cases} 1 & |t| \le T_k \\ 0 & \text{annars} \end{cases}, k=1,2$$

där $T_1 = 1$ och $T_2 = 2$. Med hjälp av Gram-Schmidts metod, hitta en ortonormerad bas för pulserna och uttryck dem med avseende på den basen. (3 p)

| **Theorems of the fourier transform** | **Function** | **Transform** |
|---|---|---|
| Linearity | $ax(t) + by(t)$ | $aX(f) + bY(f)$ |
| Time delay or time shift | $x(t-a)$ | $X(f)e^{-j2\pi fa}$ |
| Scale change | $x(at)$ | $\frac{1}{\lvert a \rvert}X(\frac{f}{a})$ |
| Conjugation | $x^*(t)$ | $X^*(-f)$ |
| Duality | $X(t)$ | $x(-f)$ |
| Frequency shift | $x(t)e^{j2\pi at}$ | $X(f-a)$ |
| Linear modulation | $x(t)\cos(2\pi at + b)$ | $\frac{e^{jb}X(f-a)+e^{-jb}X(f+a)}{2}$ |
| Differentiation | $\frac{d^n x(t)}{dt^n}$ | $(j2\pi f)^n X(f)$ |
| Integration | $\int_{-\infty}^{t} x(u)du$ | $\frac{X(f)}{j2\pi f}$ |
| Convolution | $x(t) \otimes y(t)$ | $X(f)Y(f)$ |
| Multiplication | $x(t)y(t)$ | $X(f) \otimes Y(f)$ |
| Multiplication by $t^n$ | $t^n x(t)$ | $-\frac{1}{j2\pi}\frac{d^n X(f)}{df^n}$ |

| **Fourier transforms** | **Function** | **Transform** |
|---|---|---|
| Rectangular pulse | $\mathrm{rect}(t/a)$ | $a \cdot \mathrm{sinc}(af)$ |
| Triangular pulse | $\mathrm{tria}(t/a)$ | $a \cdot \mathrm{sinc}^2(af)$ |
| Gaussian pulse | $e^{-\pi(\frac{t}{a})^2}$ | $a \cdot e^{-\pi(af)^2}$ |
| One sided exponential pulse | $e^{-t/a}u(t)$ | $\frac{a}{1+j2\pi fa}$ |
| Two sided exponential pulse | $e^{-\lvert t \rvert/a}$ | $\frac{2a}{1+(2\pi fa)^2}$ |
| Sinc pulse | $\mathrm{sinc}(at)$ | $\frac{1}{a}\mathrm{rect}(f/a)$ |
| Constant | $a$ | $a \cdot \delta(f)$ |
| Phasor | $e^{j(2\pi at+b)}$ | $e^{jb}\delta(f-a)$ |
| Cosine wave | $\cos(2\pi at + b)$ | $\frac{e^{jb}\delta(f-a)+e^{-jb}\delta(f+a)}{2}$ |
| Delayed impulse | $\delta(t-a)$ | $e^{-j2\pi fa}$ |
| Step | $u(t)$ | $\frac{\delta(f)}{2} + \frac{1}{j2\pi f}$ |

$$T_0 = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$$

$$e^{j\phi} = \cos(\phi) + j\sin(\phi)$$

$$\sin(\phi) = \frac{1}{2j}(e^{j\phi} - e^{-j\phi})$$

$$\cos(\phi) = \frac{1}{2}(e^{j\phi} + e^{-j\phi})$$

$$\sin^2\phi + \cos^2\phi = 1$$

$$\cos(\phi) = \sin(\phi - \pi/2)$$

$$\sin(\phi) = \cos(\phi + \pi/2)$$

$$\sin(\alpha)\cos(\beta) = \frac{\sin(\alpha-\beta) + \sin(\alpha+\beta)}{2}$$

$$\sin(\alpha)\sin(\beta) = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}$$

$$\cos(\alpha)\cos(\beta) = \frac{\cos(\alpha-\beta) + \cos(\alpha+\beta)}{2}$$

$$x(t) \otimes y(t) = \int_{-\infty}^{\infty} x(\lambda)y(t-\lambda)\,d\lambda = y(t) \otimes x(t)$$

$$x(t) = \sum_{k=-\infty}^{\infty} x_k e^{j2\pi k f_0 t} = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty}\Big[\alpha_k\cos(2\pi k f_0 t) + \beta_k \sin(2\pi k f_0 t)\Big]$$

$$x_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-j2\pi k f_0 t}dt$$

$$\alpha_k = 2\cdot\mathrm{Re}\{x_k\}, \quad \text{when } x(t)\in\mathbb{R}$$

$$\beta_k = -2\cdot\mathrm{Im}\{x_k\}, \quad \text{when } x(t)\in\mathbb{R}$$

$$X(f) = \mathcal{F}\Big\{x(t)\Big\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}dt$$

$$x(t) = \mathcal{F}^{-1}\Big\{X(f)\Big\} = \int_{-\infty}^{\infty} X(f)e^{j2\pi f t}df$$

$$X_k = \sum_{n=0}^{N-1} x_n e^{-j2\pi k n/N}$$

$$x_k = \frac{1}{N}\sum_{n=0}^{N-1} X_n e^{j2\pi k n/N}$$

$$f_0 = \frac{1}{N\cdot T_s} = \frac{f_s}{N}$$

$$s = \sigma + j\omega = \sigma + j2\pi f$$

$$X(s) = \mathcal{L}\Big\{x(t)\Big\} = \int_0^{\infty} x(t)\cdot e^{-st}dt$$

$$d_n = \frac{u_n}{u_1}$$

$$d_{\text{tot}} = \sqrt{\sum_{n=2}^{\infty} d_n^2}$$