ELEC-E4130 Electromagnetic fields **Midterm 1, October 19, 2023**

> Answer all questions on the provided separate answer sheets. No calculators or own papers are allowed. Submit at least one answer sheet with your name and student number. All answer sheets must be returned, including sketches and empty ones. You can keep this question sheet.

1. Answer the following briefly (2 points each). Verbal explanations are preferred.
   (a) State *Stokes' theorem* in words.
   (b) Discuss the advantages of using *vector phasors* in electromagnetics.
   (c) Describe the physical interpretation of the *Poynting vector.* What is the SI unit of this vector?
   (d) Define *Brewster angle.* When does it exist at an interface of two non-magnetic media?

2. Starting from Maxwell's equations and the constitutive relations of free space

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \qquad \nabla \cdot \mathbf{D} = \rho, \qquad \nabla \cdot \mathbf{B} = 0, \qquad \begin{cases} \mathbf{D} = \varepsilon_0 \mathbf{E} \\ \mathbf{B} = \mu_0 \mathbf{H} \end{cases}$$

   derive expressions for:
   (a) The electric field **E** of a static point source q at the origin. (4 points)
   (b) The magnetic field **H** of a static current I along the z axis in the positive direction. (4 points)

3. A uniform plane wave of angular frequency ω is incident on a prefectly conducting boundary at $z = 0$. That is, medium 1 ($z < 0$) is air and medium 2 ($z > 0$) is a perfect conductor. The incident electric field in medium 1 is given by the vector phasor

$$\mathbf{E}_i = \left[\mathbf{a}_y + j\frac{\mathbf{a}_x - \mathbf{a}_z}{\sqrt{2}}\right] E_0\, e^{-j\beta_1(x+z)/\sqrt{2}}, \qquad \beta_1 = k_0 = \omega\sqrt{\mu_0\varepsilon_0},$$

   where E_0 is a positive constant with unit V/m.
   (a) Determine the angle of incidence and polarization of the incident wave. (2 points)
   (b) Determine the vector phasor of the reflected electic field and describe the polarization of the reflected field. (6 points)

Vector identities from the back cover of Cheng's book:

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} \qquad \nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \qquad \nabla \cdot \nabla V = \nabla^2 V$$

$$\nabla(\psi V) = \psi \nabla V + V \nabla \psi \qquad \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$\nabla \cdot (\psi \mathbf{A}) = \psi \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \psi \qquad \nabla \times (\nabla V) = \mathbf{0}$$

$$\nabla \times (\psi \mathbf{A}) = \psi \nabla \times \mathbf{A} + (\nabla \psi) \times \mathbf{A} \qquad \nabla \cdot (\nabla \times \mathbf{A}) = 0$$

Plane waves:

$$\nabla \longleftrightarrow -j\mathbf{k}, \quad k = \omega\sqrt{\mu\varepsilon} = \frac{2\pi}{\lambda}, \quad \eta = \sqrt{\frac{\mu}{\varepsilon}}, \qquad u_p = \frac{\omega}{\beta}, \quad u_g = \frac{\partial \omega}{\partial \beta} = \left(\frac{\partial \beta}{\partial \omega}\right)^{-1}$$

$$\gamma = \alpha + j\beta = \omega\sqrt{\mu\varepsilon_c}, \quad \lambda = \frac{2\pi}{\beta}, \quad \delta = \frac{1}{\alpha}, \qquad n_1 \sin\theta_i = n_2 \sin\theta_t, \quad n = \sqrt{\varepsilon_r \mu_r}$$

$$\Gamma_\perp = \frac{\eta_2/\cos\theta_t - \eta_1/\cos\theta_i}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i}, \quad \tau_\perp = 1 + \Gamma_\perp, \quad \Gamma_\parallel = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}, \quad \tau_\parallel = (1 + \Gamma_\parallel)\frac{\cos\theta_i}{\cos\theta_t}$$

Nabla operations

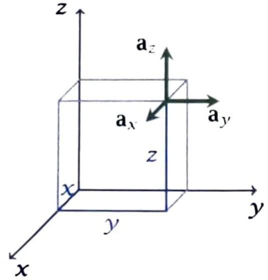
Cartesian coordinates (x, y, z)

$$\nabla V = \mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$



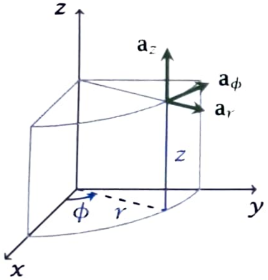
Cylindrical coordinates (r, ϕ, z)

$$\nabla V = \mathbf{a}_r \frac{\partial V}{\partial r} + \mathbf{a}_\phi \frac{1}{r}\frac{\partial V}{\partial \phi} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \mathbf{a}_r & \mathbf{a}_\phi r & \mathbf{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$



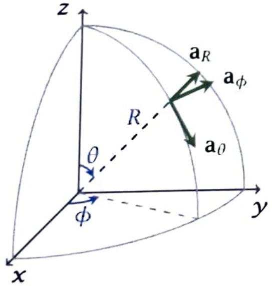
Spherical coordinates (R, θ, ϕ)

$$\nabla V = \mathbf{a}_R \frac{\partial V}{\partial R} + \mathbf{a}_\theta \frac{1}{R}\frac{\partial V}{\partial \theta} + \mathbf{a}_\phi \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2 \sin\theta}\begin{vmatrix} \mathbf{a}_R & \mathbf{a}_\theta R & \mathbf{a}_\phi R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 A_R\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta \sin\theta\right) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$



Coordinate transformations

Cartesian ↔ Cylindrical

$x = r\cos\phi$, $\quad y = r\sin\phi$, $\quad z = z$

$r = \sqrt{x^2 + y^2}$, $\quad \phi = \tan^{-1}\frac{y}{x}$, $\quad z = z$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix}$$

$$\begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Cartesian ↔ Spherical

$x = R\sin\theta\cos\phi$, $\quad y = R\sin\theta\sin\phi$, $\quad z = R\cos\theta$

$R = \sqrt{x^2 + y^2 + z^2}$, $\quad \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}$, $\quad \phi = \tan^{-1}\frac{y}{x}$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}\begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix}$$

$$\begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Cylindrical ↔ Spherical

$r = R\sin\theta$, $\quad \phi = \phi$, $\quad z = R\cos\theta$

$R = \sqrt{r^2 + z^2}$, $\quad \theta = \tan^{-1}\frac{r}{z}$, $\quad \phi = \phi$

$$\begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}\begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix}$$

$$\begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix}$$

Other useful formulas

Cartesian coordinates

$d\boldsymbol{\ell} = \mathbf{a}_x\,dx + \mathbf{a}_y\,dy + \mathbf{a}_z\,dz$

$ds_x = dy\,dz$

$ds_y = dx\,dz$

$ds_z = dx\,dy$

$dv = dx\,dy\,dz$

Cylindrical coordinates

$d\boldsymbol{\ell} = \mathbf{a}_r\,dr + \mathbf{a}_\phi r\,d\phi + \mathbf{a}_z\,dz$

$ds_r = r\,d\phi\,dz$

$ds_\phi = dr\,dz$

$ds_z = r\,dr\,d\phi$

$dv = r\,dr\,d\phi\,dz$

Spherical coordinates

$d\boldsymbol{\ell} = \mathbf{a}_R\,dR + \mathbf{a}_\theta R\,d\theta + \mathbf{a}_\phi R\sin\theta\,d\phi$

$ds_R = R^2\sin\theta\,d\theta\,d\phi$

$ds_\theta = R\sin\theta\,dR\,d\phi$

$ds_\phi = R\,dR\,d\theta$

$dv = R^2\sin\theta\,dR\,d\theta\,d\phi$

Divergence theorem $\int_V \nabla\cdot\mathbf{A}\,dv = \oint_S \mathbf{A}\cdot d\mathbf{s}$

Stokes' theorem $\int_S (\nabla\times\mathbf{A})\cdot d\mathbf{s} = \oint_C \mathbf{A}\cdot d\boldsymbol{\ell}$

Constants

$c = 299\,792\,458\ \frac{\mathrm{m}}{\mathrm{s}}$

$\mu_0 = 4\pi\times10^{-7}\ \frac{\mathrm{Vs}}{\mathrm{Am}} \approx 1.257\times10^{-6}\ \frac{\mathrm{H}}{\mathrm{m}}$

$\varepsilon_0 = \frac{1}{\mu_0 c^2} \approx 8.854\times10^{-12}\ \frac{\mathrm{As}}{\mathrm{Vm}} \quad \left(= \frac{\mathrm{F}}{\mathrm{m}}\right)$

$e \approx 1.602\times10^{-19}\ \mathrm{C}$