CS-E5795 Computational Methods in Stochastics
Exam 7.11.2023
Note: The question sheet must be returned together with the answers.
Basic function calculator is allowed.

1.

a) (4 p) Weibull distribution is defined by the CDF (cumulative distribution function)

$$F(x) = \begin{cases} 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Here, parameters $\alpha > 0$ and $\beta > 0$.
Derive and describe how you would simulate this distribution using inverse distribution method (aka inverse transform method).

b) (2 p) (This is **not** related to the above part a).) What are the resulting distributions of positions after a large number of steps when at constant time intervals random walkers starting at the initial position $x = 0$...

(i) ... either take two steps right (x increases) or one step left (x decreases) with equal probability?

(ii) ... either move right the distance of two steps multiplied by their current position or move left the distance of one step multiplied by their current position?

You need not give any exact parameter values for either distribution; however, make a general statement about the mean.

2. A Markov chain $X_0, X_1, X_2, X_3, \ldots, X_n, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}.$$

$e^0 = 1$
$0 = \ln(1)$
$e^x = y$
$x = \ln(y)$

The states are labelled as $S = \{0, 1, 2\}$.

a) (1 p) What is the probability distribution of the values of X_5 under the condition that $X_4 = 1$?

b) (2 p) What is the probability distribution of the values of X_2 under the condition that $X_0 = 0$?

c) (2 p) After having simulated this process, you decide that it is sufficient to update states only for every three steps. Accordingly, you simulate a Markov chain, where each step corresponds to three steps in the original Markov chain, so that the states in this new process are $Y_0 = X_0, Y_1 = X_3, Y_3 = X_6, \ldots$ What is the transition probability matrix for this Markov chain?

d) (1 p) In an assignment radioactive decay was simulated. Starting from the differential equation for the number of undecayed nuclei $\frac{dN(t)}{dt} = -\lambda N(t)$, where λ is constant, this was shown to be an inhomogeneous Poisson process. Is the radioactive decay an inhomogeneous Poisson process in the asymptotic limit $N \to \infty$? A very brief ($\approx$ one sentence) justified answer suffices.

3.

a) (3 p) Markov chain in discrete time t: Denoting the transfer kernel of a Markov process as $P(x, y) = P(\theta^{(t+1)} = y|\theta^{(t)} = x)$ and probability densities as $P\left(\theta^{(t)} = x\right) = \pi^{(t)}(x)$, derive the transition kernel for the reversed chain $P_t^*(x, y) = P(\theta^{(t)} = y|\theta^{(t+1)} = x)$. Under what condition is this reversed chain homogeneous?

b) (3 p) Derive in as much detail as you can the potential function $U(q)$ to be used in the Hamiltonian Monte Carlo (HMC) method for simulating the normal distribution $N(3, 2)$, whose probability density function is $\pi(q) = \frac{1}{\sqrt{8\pi}} e^{-\frac{(q-3)^2}{8}}$, $-\infty < q < \infty$. In the end, give the potential in the minimal form for using in HMC. Justify this minimal form. If you cannot derive $U(q)$, you can try to briefly justify it.

4.

a) (2 p) Simulating a stochastic process, you find that the probability density function (pdf) you sample roughly preserves its form while the mean value increases in time. Can the Markov chain used to simulate this process be reversible? Can detailed balance hold for this Markov chain? Verbal justification suffices. Of course, you can include maths if you can.

b) (4 p) Describe with equations and diagrams in phase space coordinates the two steps comprising Hamiltonian Monte Carlo (HMC) method. When appropriate, explain reasons for the details in the procedure. What is the motivation for using HMC? In this method, how do you take into account finite support of a target distribution?

0,7 · 6 · 4

6 5 = 11

= 0,1 · 7 6 4

= 0,1 · 42 · 4

6 5 1 2, 3/4

= 16,8

17 – 17