Instructions: Answer as many questions as possible. Each subquestion (labelled with letters) carries equal weight, and is worth a maximum mark of 6 points.

It is only permitted to bring to the exam room basic writing material and a scientific calculator.

1. Let $A = \begin{bmatrix} 3 & 1 \\ 9 & -3 \end{bmatrix}$. TYPO $\begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}$

   (a) Prove that
   $$e^{tA} = \begin{bmatrix} 1+3t & -t \\ 9t & 1-3t \end{bmatrix}.$$
   Hint. Show first that $A^2 = 0$. What does it imply?

   (b) Using part (a) or otherwise, compute the solution to the following system of ODEs:
   $$\mathbf{y}'(t) = A\mathbf{y}(t); \qquad \mathbf{y}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

2. Let
   $$B = \begin{bmatrix} 1 & 0 & 1-y \\ 0 & 1-y & 1 \\ 1-y & 0 & 1 \end{bmatrix}$$
   where $y \in \mathbb{R}$ is a parameter.

   (a) Discuss for what values of y the linear system $B\mathbf{x} = \mathbf{b}$ has a solution for every right hand side $\mathbf{b} \in \mathbb{R}^3$.

   (b) Give the general definition of $R(M)$, the range of a matrix M. Then, compute a basis for $R(B)$ both in the special case where $y = 2$ and in the special case where $y = 0$.

3. Let
   $$C = \begin{bmatrix} 2 & -4 \\ 0 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}, \qquad \mathbf{d} = \begin{bmatrix} 0 \\ -18 \\ 18 \\ 0 \end{bmatrix}.$$

   (a) Using the Gram-Schmidt algorithm or otherwise, compute a QR decomposition of the matrix C.

   (b) Prove that $N(C) = \{\mathbf{0}\}$, and explain how to use this fact to determine the number of solutions to the least square problem
   $$\min_{\mathbf{x}\in\mathbb{R}^2} \|C\mathbf{x} - \mathbf{d}\|.$$
   Then, using part (a) or otherwise, compute all the solutions to such least square problem.

4. Let $M \in \mathbb{C}^{n\times n}$ be a square complex matrix satisfying $|M_{ij}| < 1$ for all $1 \leq i, j \leq n$.

   (a) Give the definition of an eigenvalue and an eigenvector of the matrix M. Then, prove that if λ is an eigenvalue of M then $|\lambda| < n$. *Hint: Use the property that the modulus of every element of M is less than* 1.