The exam consists of 4 problems, each worth 6 points.
You are allowed to use a **scientific calculator (funktiolaskin)** in the exam; graphic and symbolic calculators are forbidden. You are also allowed to use a **handwritten memory aid sheet** of size A4, with text only on one side and with your name and student number in the upper right corner.

1 Consider processes on state space $S = \{-1, 0, 1\}$, and consider the following matrices:

$$M_1 = \begin{bmatrix} 0.2 & 0.5 & 0.2 \\ 0.8 & 0 & 0.2 \\ 0 & 0.5 & 0.6 \end{bmatrix}, \quad M_2 = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -3 & 2 \\ 0 & 2 & -2 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0.2 & 0.8 & 0 \\ 0.5 & 0 & 0.5 \\ 0.2 & 0.2 & 0.6 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} -1 & 0 & 2 \\ 2 & -3 & 2 \\ 0 & 2 & -1 \end{bmatrix}, \quad M_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}$$

Justify your answers to the following questions briefly.

(a) **(1p.)** Which of the matrices are transition matrices?

(b) **(1p.)** Which of the transition matrices have unique invariant distributions?

(c) **(2p.)** Take one of the transition matrices whose corresponding Markov chain $X = (X_0, X_1, X_2, \ldots)$ started at state $X_0 = 0$ has a limiting distribution. Argue why the limiting distribution exists and find the limiting distribution.

(d) **(1p.)** Which of the matrices are generator matrices?

(e) **(1p.)** Take one of the generator matrices. Let $Z = (Z_t)_{t\in[0,\infty)}$ be the corresponding continuous-time Markov chain. Given that $Z_0 = 0$, what is the probability that the first jump of the process goes to state 1?

2 Developers of two computer algorithms "muGo" and "nuGo" want to investigate which one is stronger in the game of go. They do so by letting muGo and nuGo play against each other repeatedly. At each round, muGo wins with probability 0.6 independently from other games. Denote the number of games after $t = 0, 1, 2, \ldots$ rounds won by muGo and nuGo by M_t and N_t respectively, and let $X_t = M_t - N_t$.

Justify your answers to the following questions.

(a) **(2p.)** Model the process $X = (X_t)_{t\in\mathbb{N}_0}$ as a Markov chain. What is its state space? Write down its transition matrix and draw its transition diagram.

(b) **(2p.)** Does X have an invariant distribution? If yes, how many?

(c) **(2p.)** The developers decide to let muGo and nuGo to compete in a tournament. The first algorithm to win two games more than the other is declared as the winner of the tournament. What is the probability that nuGo wins the tournament?

3 A new disease spreads in the city of Espoo. Epidemiologists have decided to model the epidemic using a branching process $X = (X_t)_{t \in \mathbb{N}_0}$ on an infinite population size.

In the morning of day zero, one person is infectious and the rest are susceptible for the disease. During each day, every infectious person contacts two susceptible persons, and in the evening the infectious person recovers. Each contact is infectious with probability 2/3, independently of others. If a contact is infectious, then the targeted susceptible person gets an infection and becomes infectious in the morning of the next day.

Justify your answers to the following questions.

(a) **(1p.)** Write down the probability generating function for the offspring distribution for the number of people who directly receive the infection from the initially infected person.

(b) **(1p.)** Compute the expected number of people who directly receive the infection from the initially infected person.

(c) **(1p.)** What is the probability that there is at least one infectious person at day 2?

(d) **(1p.)** What is the probability that the epidemic eventually dies out, so that on some day there will be no infectious people in the population?

(e) **(2p.)** Scientists have developed a vaccine against the disease. In testing, 90% of vaccinated people obtained full immunity of the disease, while for the remaining 10% the vaccine had no effect. If they vaccinate $\lambda\%$ of the population, then with probability $\frac{\lambda}{100}$ each contact made by an infectious person is with a vaccinated person, independently of others. At least what percentage of the population has to be vaccinated to ensure that the epidemic eventually dies out?

4 Romeo and Juliet are trying to contact each other using a radio phone having three possible channels denoted by 1, 2, and 3. Unfortunately, they forgot to agree on the channel. As a result, both Romeo and Juliet cycle through the channels in the order $1 \to 2 \to 3 \to 1$. They spend independent and identically distributed exponential time at the channel, and change the channel on average once in every 5 seconds.

Denote by X_t and Y_t the radio channels of Romeo and Juliet, respectively, at time t (where the unit of time is seconds).

(a) **(1p.)** Argue why $X = (X_t)_{t\in[0,\infty)}$ and $Y = (Y_t)_{t\in[0,\infty)}$ are independent continuous-time Markov chains and draw their transition diagrams.

(b) **(2p.)** Denote by $N_R(t)$ and $N_J(t)$ the number of times Romeo and Juliet, respectively, have changed the channels during t seconds, and let $N(t) = N_R(t) + N_J(t)$. Find the distribution of $N(t)$.

(c) **(1p.)** Let T_k be the k:th jump time of N. Let $\theta_1, \theta_2, \ldots$ be random variables defined as follows:

$$\theta_{2k} = \begin{cases} 1, & \text{Romeo changed channel at time } T_{2k}, \\ 0, & \text{Juliet changed channel at time } T_{2k}, \end{cases}$$

$$\theta_{2k+1} = \begin{cases} 1, & \text{Juliet changed channel at time } T_{2k+1}, \\ 0, & \text{Romeo changed channel at time } T_{2k+1}. \end{cases}$$

Argue why $\theta_1, \theta_2, \ldots$ are independent and identically distributed.

(d) **(2p.)** Romeo and Juliet successfully contact each other when they end up having their radios in the same channel. Given that $X_0 = 1$ and $Y_0 = 2$, what is the expected time for that to happen?