Instructions: Answer as many questions as possible. Each subquestion (labelled with letters) carries equal weight, and is worth a maximum mark of 6 points.

It is only permitted to bring to the exam room basic writing material and a scientific calculator.

1.

$$A = \begin{bmatrix} 0 & 1 \\ 1/\sqrt{5} & 0 \\ -2/\sqrt{5} & 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 7 \\ 2\pi \\ \pi \end{bmatrix}$$

(a) Say whether the following claim is true or false: $A = A \cdot I_2$ is a QR decomposition of $A$. Justify your answer.

(b) Using part (a) or otherwise, describe the set of solutions to the least squares problem $\min_{\mathbf{x} \in \mathbb{R}^2} \|A\mathbf{x} - \mathbf{b}\|_2$.

2. Let

$$B = \begin{bmatrix} y-1 & 0 & 1 \\ 1 & y+1 & 1 \\ -1 & 0 & y+1 \end{bmatrix}$$

where $y \in \mathbb{R}$ is a parameter.

(a) Prove that the eigenvalues of $B$ are $y$ and $y+1$, and compute an eigenvector associated with each of them.

(b) Say whether the following claim is true or false: The algebraic multiplicity of the eigenvalue $y$ is equal to the geometric multiplicity of the eigenvalue $y$. Justify your answer.

(c) Say for which values of $y$ the linear system $B\mathbf{x} = \mathbf{c}$ has a unique solution for every possible right hand side $\mathbf{c} \in \mathbb{R}^3$.

3. The *Frobenius norm* of a matrix $C \in \mathbb{R}^{m \times n}$ is defined as $\|C\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |C_{ij}|^2}$. *Hint: In solving the next questions, you can freely use the following property. For every pair of orthogonal matrices $Q \in \mathbb{R}^{m \times m}$, $Z \in \mathbb{R}^{n \times n}$ and every matrix $C \in \mathbb{R}^{m \times n}$ it holds $\|C\|_F = \|QCZ\|_F$.*

(a) Prove that, if $m \geq n$ and the singular values of $C$ are $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$, then

$$\|C\|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2} = \sqrt{\sum_{i=1}^{n} \sigma_i^2}.$$

(b) Using the statement of item (a) or otherwise, prove that if $m \geq n$ then every $C \in \mathbb{R}^{m \times n}$ satisfies both $\|C\|_2 \leq \|C\|_F$ and $\|C\|_F \leq \sqrt{n}\|C\|_2$. Moreover, give an example of a matrix $A \in \mathbb{R}^{2 \times 2}$ such that $\|A\|_F = \|A\|_2$, and an example of a matrix $B \in \mathbb{R}^{2 \times 2}$ such that $\|B\|_F = \sqrt{2}\|B\|_2$.

(c) Let $U \in \mathbb{R}^{n \times n}$. Using the statements of items (a) and (b) or otherwise, prove that $\|U\|_2 = \|U\|_F$ if and only if $\dim R(U) = 1$.