MS-E1461 Hilbert spaces **(Aalto University, Turunen / Vavilov)**
Examination on Monday 16.10.2023, at 9:00-12:00

Points also for good effort! No literature, no calculators, etc. If you refer to any results in the course, please name them.

1. Let v_1, v_2 be two linearly independent vectors in a Hilbert space.

   (a) Find an orthonormal basis for vector subspace $Z = \mathrm{span}(\{v_1, v_2\})$.

   (b) Find the formulas for the orthogonal projection P onto Z, and for the orthogonal projection Q onto the orthogonal complement of Z.

   (c) Sketch also a picture (or pictures) how these vectors and projections look like in case of a 3-dimensional real Hilbert space.

2. Let $L^2([0,1])$ denote Lebesgue's space of absolutely square-integrable functions (No need to go into details of measure and integration theory here). Let $e_k(x) := \mathrm{e}^{\mathrm{i}2\pi xk}$.

   (a) Explain what we mean when saying that $(e_k)_{k\in\mathbb{Z}}$ is an orthonormal basis. What is the Fourier series representation of $u \in L^2([0,1])$ then?

   (b) Using the orthonormal basis in (a), construct a countable orthonormal basis for $L^2(\mathbb{R})$, which is Lebesgue's space of the absolutely square-integrable functions on the real line.

3. Formulate and prove the Fréchet–Riesz Representation Theorem for bounded linear functionals in Hilbert spaces.

4. Let $H := \ell^2(\mathbb{Z}^+)$, i.e. Hilbert space of the absolutely square-summable functions $u : \mathbb{Z}^+ \to \mathbb{C}$, and let

$$Au(x) := \sum_{y=1}^{\infty} 2^{-(x+y)}\, u(y) \tag{1}$$

   (a) Show that (1) defines a compact linear operator $A : H \to H$. Show that A is self-adjoint. Find the operator norm $\|A\|$.

   (b) Diagonalize operator A.