MS-E1652 Computational methods for differential equations

Course exam and Exam, 9:00–12:00, December 4, 2023 **Hyvönen/Khuat**

Calculators or other extra material are not allowed.

The grade based on the combination of the course exam and the exercise points is deduced by accounting for the four solutions that produce the most points; if a student, e.g., gets 6, 2, 4, 4 and 5 points from the individual exam problems, then the number of exam points considered in connection to the exercise points is 19 = 6 + 4 + 4 + 5. The grade based on the mere final exam is deduced by accounting for all five solutions. The better of the two alternatives will correspond to the final grade.

1. Consider the *linear multistep method* (LMM)

$$x_{j+1} - x_j = \frac{1}{3}h\big(2f_{j+1} + f_j\big) \qquad j = 0, 1, 2, \dots \,. \tag{1}$$

   Consider the following questions/tasks. Justify your answers.

   (a) Is (1) an explicit or an implicit LMM?

   (b) Prove that (1) is consistent of order $p = 1$, but not of order $p = 2$.

   (c) Is (1) zero-stable?

   (d) Why is zero-stability an indispensable property of any functional LMM?

2. Prove that the LMM (1) is A-stable. In other words, prove that its region of absolute stability $\mathcal{R}$ contains the open left half of the complex plain $\mathbb{C}_- = \{z \in \mathbb{C} \,|\, \mathrm{Re}\, z < 0\}$.

3. Consider the *Runge–Kutta* (RK) method

$$\begin{aligned} x_{j+1} &= x_j + \frac{1}{3}h(k_1 + 2k_2), \\ k_1 &= f(t_j, x_j), \\ k_2 &= f\Big(t_j + h, x_j + \frac{h}{3}(k_1 + 2k_2)\Big). \end{aligned} \tag{2}$$

   (a) Is (2) an explicit or an implicit RK method? Why?

   (b) Deduce the stability function corresponding to (2)?

   (c) Is the method (2) A-stable? Justify your answer. (cf. Problem 2)

   (d) What do you expect to be the consistency order of (2)? You can base your argument, e.g., on the stability function. Justify your answer. (Hint: $\sum_{j=0}^{\infty} q^j = (1-q)^{-1}$ for $|q| < 1$.)

4. Consider the second order initial value problem

$$x''(t) = x(t), \qquad x(0) = 1, \quad x'(0) = -1. \tag{3}$$

   (a) Solve (3) and, in particular, note that $\lim_{t\to\infty} x(t) = 0$.

   (b) Write (3) in the form

$$y'(t) = Ay(t), \qquad y(0) = y_0, \tag{4}$$

   where $y(t) = [x(t), x'(t)]^{\mathrm{T}}$. That is, determine $A \in \mathbb{R}^{2\times 2}$ and $y_0 \in \mathbb{R}^2$ such that the initial value problems (3) and (4) are equivalent.

   (c) Apply Euler's method to the initial value problem (4) with A and y_0 as specified in part (b). For which values of the step size $h > 0$ does it hold that

$$\lim_{j\to\infty} y_j = 0 \ ?$$

   Here $y_j \approx y(t_j)$, with $t_j = jh$, is the sequence of numerical approximations produced by Euler's method.

5. When the initial/boundary value problem for the heat equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & x \in (0,1),\ t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = g(x), & x \in (0,1), \end{cases}$$

is spatially discretized by the standard central second order difference approximation, one ends up with the following initial value problem:

$$U'(t) = A\,U(t), \qquad U(0) = G, \tag{5}$$

for all $t \geq 0$. Here, $G = [g(x_1), \ldots, g(x_m)]^{\mathrm{T}}$ and $U(t) \approx [u(x_1,t), \ldots, u(x_m,t)]^{\mathrm{T}}$, with $x_j = jh$ and $h = 1/(m+1)$ being the mesh parameter.

(a) What does the difference matrix $A \in \mathbb{R}^{m\times m}$ look like? (It is enough to remember/reason the structure of A — you need not present an actual proof.)

(b) Introduce some consistent numerical method for solving (5). Let $\delta > 0$ be the time step size and denote by U_k the approximation of $U(k\delta)$ for $k = 0, 1, 2, \ldots$.

(c) For which $\delta > 0$ is your method (for sure) absolutely stable, that is,

$$\lim_{k\to\infty} U_k = 0 \in \mathbb{R}^m$$

for any $G \in \mathbb{R}^m$? Justify your answer.

You can use the following facts without proving them:

1. The eigenvalues of $A \in \mathbb{R}^{m\times m}$ satisfy $0 > \lambda_1 > \lambda_2 > \cdots > \lambda_{m-1} > \lambda_m > -4/h^2$.
2. For any matrix $B \in \mathbb{R}^{m\times m}$, it holds that

$$\lim_{k\to\infty} B^k = 0 \in \mathbb{R}^{m\times m},$$

if and only if the eigenvalues of B satisfy $|\mu_j| < 1$, $j = 1, \ldots, m$.