MS-A0111 – Differential and Integral Calculus 1 | **Exam**
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Remember to write your name and student number on the solutions you return. And note that this sheet has a second page! You can solve the problems in any order. You are not allowed to use a calculator, tables or notes.

Every problem carries an equal weight. Similarly, every part of a problem carries an equal weight.

Explain the reasoning behind your solutions, do not just write the final result.

Problem 1. Compute the following limits:

$$\text{(a)} \quad \lim_{x\to 0} \frac{1-\cos(x^2)}{\sin(2x)} \qquad\qquad \text{(b)} \quad \lim_{x\to 0} \frac{e^{4x}-1-4x}{1-\cos(4x)}.$$

You may use any method we covered in the course: famous limits, L'Hôpital's rule, Taylor polynomials with error in big-O notation.

Problem 2. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{x+1}{e^x}$.

(a) For which values of x is $f(x)$ positive/negative/zero?

(b) Compute the derivative of f. Where is it positive/negative/zero?

(c) Compute the second derivative of f. Where is it positive/negative/zero?

(d) Compute the limits $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to+\infty} f(x)$.

(e) Use the information you found to scketch the graph of f.

Problem 3. Compute the following integrals:

$$\text{(a)} \quad \int xe^{-6x}\,dx \qquad\qquad \text{(b)} \quad \int_0^{2\sqrt{2}} \frac{3x^3}{\sqrt{x^2+1}}\,dx.$$

Hint: In (b), use the substitution $u = x^2 + 1$.

Problem 4. Consider the function $f(x) = e^{2x} + \sin(\pi x)$.

(a) Write the second-order Taylor polynomial for f at $a = 0$.

(b) Write the second-order Taylor polynomial for f at $a = 1$.

Warning: In (b), it's not enough to take the Taylor polynomial about 0, found in (a), and plug in something other than x in the place of x. Instead, work it out from the definition. Leave all values such as π or e^2 as they are, do not approximate them.

Problem 5. Find all solutions to the differential equation

$$y'' - 3y' - 10y = 0.$$

Among all the solutions, find the one that satisfies $y(0) = y'(0) = 7$.

Problem 6. Find all solutions to the differential equation

$$y' = x^2 y^2.$$

Among all the solutions, find the one that satisfies $y(0) = 1$.

Formulas.

Some Maclaurin approximations, with error written in big-O notation:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + O(x^{n+1}) \qquad \text{as } x \to 0$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + O(x^{2n+2}) \qquad \text{as } x \to 0$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + O(x^{2n+3}) \qquad \text{as } x \to 0$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \frac{x^n}{n} + O(x^{n+1}) \qquad \text{as } x \to 0$$

Some special values of trigonometric functions:

α	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
$\sin(\alpha)$	$-1/\sqrt{2}$	$-1/2$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0
$\cos(\alpha)$	$1/\sqrt{2}$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0	-1