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**Exam, Monday, December 4, 2023, 09:00 - 12:00**
Complex Analysis, MS-C1300

Motivate your answers. Only giving answers gives no points. No calculators or books are allowed. **Good luck!**

(1) (a) Find the Taylor series of

$$f(z) = \frac{1}{(2+z)^2}$$

around $z_0 = 0$. Determine the radius of convergence $\rho$ for the series. (2p)

(b) Find the Taylor series of

$$f(z) = \frac{z}{2z^2 - 3z + 1}$$

around $z_0 = 0$. Determine the radius of convergence $\rho$ for the series. (*Hint:* Partial fractions help.) (2p)

(c) Find the Laurent series of

$$f(z) = \frac{e^z}{(2z - i\pi)^2}$$

in $\{z \in \mathbb{C}; 0 < |z - \frac{i\pi}{2}|\}$. (2p)

(2) Let $\gamma(t) = e^{it} + 1$, for $0 \le t \le 2\pi$. Calculate:

winding

(a)

$$\int_\gamma \frac{e^z + z}{z+1}\, dz$$

(2p)

(b)

$$\int_\gamma \frac{e^z + z}{z-1}\, dz$$

(2p)

(c)

$$\int_\gamma \frac{e^z + z}{(2z-1)^2}\, dz.$$

(2p)

(3) Assume that $a > 0$ is a strictly positive real number. Let

$$f(z) = \frac{e^{iz}}{z^2 + a^2}.$$

(a) Find the poles of $f$ and determine their order. Calculate the residues of $f$ at those poles. (3p)

(b) Calculate

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx.$$

(3p)

(4) Let $\Delta(0,1) = \{z \in \mathbf{C} : |z| < 1\}$ and $f : \Delta(0,1) \to \mathbf{C}$ be analytic. Assume that $f(0) = 0$ and $|f(z)| \le M$ for all $z \in \Delta(0,1)$. Show that $|f(z)| \le M|z|$ for all $z \in \Delta(0,1)$. (*Hint:* Study the function

$$g(z) = \begin{cases} f(z)/z \text{ when } z \neq 0 \\ f'(0) \text{ when } z = 0. \end{cases}$$

Is this function analytic? How big can $|g(z)|$ be on $\Delta(0,1)$? Use this information to prove the result. This result is called Schwarz's lemma.) (6p)

## Useful formulas

**- Cauchy–Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

**- Cauchy's Integral Formula**

$$n(\gamma, z_0) f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{k+1}} dz$$

**- Residue for a pole of order $m$**

$$\mathrm{Res}(z_0, f) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]$$

**- Some Taylor series**

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \text{ when } |z| < 1$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \text{ when } z \in \mathbf{C}$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \text{ when } z \in \mathbf{C}$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \text{ when } z \in \mathbf{C}$$