Aalto University

Partial Differential Equations

Course exam 11.12.2023

Course exam (50%)
No calculators, notes or other equipment

Solve all five problems. Each is worth 6 points.
Some formulas are given in the end of the exam paper.

Problem 1: Let $f \in C^2(\mathbb{R}^2)$ and $a, b \in \mathbb{R}$. Show that $u(x,y) = f(ax+by)$ is a solution to the Monge-Ampère equation $u_{xx}u_{yy} - u_{xy}^2 = 0$.

Problem 2: Let $\Omega = \{x \in \mathbb{R}^2, : \ |x| = 1\}$ be the one-dimensional unit ring. Consider the heat equation $u_t = \Delta u$ on $\Omega \times (0,T)$ with the initial temperature given by a function $g \in C(\Omega)$. How can this problem be solved by separation of variables? A brief list of the main steps of the argument is enough.

Problem 3: Let $\Omega \subset \mathbb{R}^n$ be a bounded and connected set with smooth boundary and $f, g \in C(\partial\Omega)$. Consider the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Show that this problems has at most one solution $u \in C^2(\Omega)$ for example by using Green's identities. (If you use comparison principle, you should prove it as well.)

Problem 4: Give solution formulas to the problems below in terms of the fundamental solution of the heat equation Φ and the initial data. It is enough to give the explicit formulas without any proofs.

(a) The Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & x \in \mathbb{R}^n, \end{cases}$$

where $g \in C_0^\infty(\mathbb{R}^n)$.

(b) The nonhomogeneous Cauchy problem with zero boundary values

$$\begin{cases} u_t - \Delta u = f & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = 0 & \text{in} \quad \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where $f \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$.

(c) The nonhomogeneous Cauchy problem with general boundary values

$$\begin{cases} u_t - \Delta u = f & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & x \in \mathbb{R}^n, \end{cases}$$

where $f \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$ and $g \in C_0^\infty(\mathbb{R}^n)$.

Problem 5: Consider the initial and boundary value problem on the first quadrant

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad \text{in} \quad \mathbb{R}_+ \times (0,\infty), \\ u(x,0) = \sin(2x), \quad u_t(x,0) = \sin(x) \qquad x \in \mathbb{R}_+, \\ u = 0 \quad \text{on} \quad \{x=0\} \times (0,\infty). \end{cases}$$

Solve this problem by using the method of odd reflection and d'Alembert's formula.

Some formulas that might be useful:

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \quad 0 < r < \infty, \quad -\pi \le \theta < \pi.$$

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega u \Delta v \, dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} u \, dS.$$

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, & x \in \mathbb{R}^n, \quad t > 0, \\ 0, \quad x \in \mathbb{R}^n, \quad t \le 0. \end{cases}$$

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy.$$

$$v(x,t) = \frac{1}{|\partial B(x,t)|} \int_{\partial B(x,t)} \left(th(y) + g(y) + \nabla g(y) \cdot (y-x)\right) \, dS(y).$$

$$u(x,t) = \int_0^t u(x,t;s)\, ds, \quad x \in \mathbb{R}^n, \quad t > 0,$$

where $u(x,t;s)$, with $0 < s < t$, is a solution the problem

$$\begin{cases} \frac{\partial u}{\partial t}(x,t;s) - \Delta u(x,t;s) = 0, \quad x \in \mathbb{R}^n, \quad t > s, \\ u(x,s;s) = f(x,s), \quad x \in \mathbb{R}^n. \end{cases}$$