ELEC-E8101 Digital and Optimal Control

Final Exam (05.12.2023) – Solution

1. Consider the linear, time-invariant, discrete-time state-space system

$$x[k+1] = \begin{pmatrix} 0 & a_1 \\ 1 & a_2 \end{pmatrix} x[k] + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u[k]$$
$$y[k] = \begin{pmatrix} 1 & 0 \end{pmatrix} x[k].$$

   (a) Derive the characteristic polynomial of the system matrix Φ! [0.5p]

   (b) Use Jury's stability criterion to determine the stability conditions! [1p]

   (c) Assume $a_2 = 1$. Illustrate graphically for which choices of a_1 the system is stable! [1.25p]

   (d) Derive the controllability matrix and check for which choices of a_1, a_2 the system is reachable! [0.5p]

   (e) Derive a pole placement controller that places both poles of the system at 0.5! Write the controller gains as functions of a_1 and a_2. [1.25p]

   (f) Could we have also placed the poles at different locations or are we restricted in where to locate the poles? Why? [0.25p]

   (g) Derive the observability matrix and check for which choices of a_1, a_2 the system is observable! [0.5p]

   (h) Derive a deadbeat observer for the system! Write the observer gains as functions of a_1 and a_2. (The design with delay is sufficient, i.e., no need to derive the Luenberger observer.) [1.25p]

   (i) What is the main characteristic of a deadbeat controller or observer? [0.25p]

Solution. (a) The characteristic polynomial is derived by computing the eigenvalues of Φ:

$$\begin{aligned} \chi(\lambda) &= \det(\lambda I - \Phi) \\ &= \det \begin{pmatrix} \lambda & -a_1 \\ -1 & \lambda - a_2 \end{pmatrix} \\ &= \lambda^2 - a_2\lambda - a_1. \end{aligned}$$

b) The derivations for Jury's stability criterion are shown below.

$a_0 \quad a_1 \quad a_2$	1	$-a_2$	$-a_1$	
$a_2 \quad a_1 \quad a_0$	$-a_1$	$-a_2$	1	$b_2 = \frac{-a_1}{1} = -a_1$
$a_0^1 \quad a_1^1$	$1-(a_1)^2$	$-a_2(1+a_1)$		
$a_1^1 \quad a_0^1$	$-a_2(1+a_1)$	$1-(a_1)^2$		$b_1 = \frac{-a_2(1+a_1)}{1-(a_1)^2} = \frac{-a_2}{1-a_1}$
a_0^0	$1-(a_1)^2 - \frac{(a_2)^2(1+a_1)}{1-a_1}$			

Thus, the stability conditions are

$$1-(a_1)^2 > 0$$
$$1-(a_1)^2 - \frac{(a_2)^2(1+a_1)}{1-a_1} > 0.$$

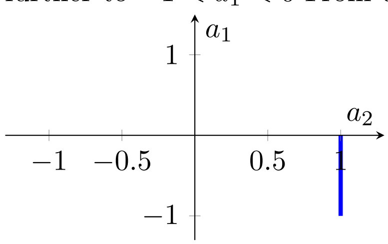
c) Factoring out the first inequality, we find

$$(1 - a_1)(1 + a_1) > 0.$$

From this inequality, we can already conclude that $-1 < a_1 < 1$. For the second inequality, we find

$$\begin{aligned} 1 - (a_1)^2 - \frac{(1 + a_1)}{1 - a_1} &= \frac{(1 - a_1)^2(1 + a_1) - (1 + a_1)}{1 - a_1} \\ &= \frac{(1 + a_1)[(1 - a_1)^2 - 1]}{1 - a_1} \\ &= \frac{(1 + a_1)((a_1)^2 - 2a_1)}{1 - a_1}. \end{aligned}$$

If $(1 - a_1)(1 + a_1) > 0$, then also $\frac{1+a_1}{1-a_1} > 0$. Thus, the two conditions are equivalent, and we are left with $(a_1)^2 - 2a_1 > 0$ or $(a_1)^2 > 2a_1$. This reduces the admissible region even further to $-1 < a_1 < 0$ From this, we can draw the plot:



d) The controllability matrix is

$$W_c = \begin{pmatrix} \Gamma & \Phi\Gamma \end{pmatrix} = \begin{pmatrix} 0 & a_1 \\ 1 & a_2 \end{pmatrix}.$$

For the system to be reachable, the controllability matrix must have full rank. This is the case if $a_1 \neq 0$.

e) We first derive the characteristic polynomial of the closed-loop system

$$\begin{aligned} \chi(\lambda) &= \det(\lambda I - (\Phi - \Gamma L)) \\ &= \det \begin{pmatrix} \lambda & -a_1 \\ -1 + \ell_1 & \lambda - a_2 + \ell_2 \end{pmatrix} \\ &= \lambda^2 + (\ell_2 - a_2)\lambda - a_1 + a_1\ell_1. \end{aligned}$$

The desired polynomial given the poles is

$$\chi(\lambda) = (\lambda - 0.5)^2 = \lambda^2 - \lambda + 0.25.$$

Thus, we find

$$\ell_2 = a_2 - 1$$

and

$$\begin{aligned} a_1\ell_1 - a_1 &= 0.25 \\ \ell_1 &= \frac{0.25 + a_1}{a_1}. \end{aligned}$$

f) Given $a_1 \neq 0$, the system is reachable. Thus, we can place the poles at arbitrary locations. Poles with an absolute value larger than one would be unstable, but also this would not prevent us from placing them there.

g) The observability matrix is

$$W_\mathrm{o} = \begin{pmatrix} C \\ C\Phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a_1 \end{pmatrix}.$$

For the system to be observable, the observability matrix must have full rank. This is the case if $a_1 \neq 0$.

h) We first derive the characteristic polynomial

$$\begin{aligned} \chi(\lambda) &= \det(\lambda I - (\Phi - KC)) \\ &= \det \begin{pmatrix} \lambda + k_1 & -a_1 \\ k_2 - 1 & \lambda - a_2 \end{pmatrix} \\ &= (\lambda + k_1)(\lambda - a_2) + a_1(k_2 - 1) \\ &= \lambda^2 + (k_1 - a_2)\lambda + a_1(k_2 - 1) - a_2 k_1. \end{aligned}$$

Since we consider a deadbeat controller, the desired characteristic polynomial is

$$\chi(\lambda) = \lambda^2.$$

Thus, we find

$$k_1 = a_2$$

and

$$\begin{aligned} a_1(k_2 - 1) - a_2 k_1 &= 0 \\ k_2 &= \frac{a_1 + a_2 k_1}{a_1} \\ &= \frac{a_1 + (a_2)^2}{a_1} \\ &= 1 + \frac{(a_2)^2}{a_1}. \end{aligned}$$

i) A deadbeat controller brings the system to the desired state in at most n time-steps, where n is the dimensionality of the state.

2. Consider the scalar system

$$x[k+1] = x[k] + u[k] + v[k]$$
$$y[k] = x[k] + w[k].$$

The noise signals v, w are white, uncorrelated, and have covariance r_1 and r_2, respectively. We use the standard quadratic cost and call the cost associated with the state q_{x} and the cost associated with the inputs q_{u} (since r is already used for the covariance), where $q_{\mathrm{x}}, q_{\mathrm{u}} > 0$.

(a) Derive the steady-state covariance P_∞ and the steady-state Kalman gain K_∞ for the system starting from the general formulas for $K[k]$ and $P[k]$!
Hint: for a steady-state equation, we must have $P[k+1] = P[k]$. [1.25p]

(b) What can you say about the mean and the variance of the prediction error when using the Kalman filter? (No calculation needed, just general properties.) [0.5p]

(c) Compute the steady-state LQR for the system, i.e., determine the gain L!
Hint: for the steady-state LQR gain, we have [1p]

$$L = (\Gamma^{\mathrm{T}} S \Gamma + q_{\mathrm{u}})^{-1} \Gamma^{\mathrm{T}} S \Phi$$

where

$$S = \Phi^{\mathrm{T}} [S - S\Gamma(\Gamma^{\mathrm{T}} S \Gamma + q_{\mathrm{u}})^{-1} \Gamma^{\mathrm{T}} S] \Phi + q_{\mathrm{x}}.$$

(d) Assume

$$L = \frac{\frac{q_{\mathrm{x}}}{2} + \sqrt{\frac{q_{\mathrm{x}}^2}{4} + q_{\mathrm{x}} q_{\mathrm{u}}}}{\frac{q_{\mathrm{x}}}{2} + \sqrt{\frac{q_{\mathrm{x}}^2}{4} + q_{\mathrm{x}} q_{\mathrm{u}}} + q_{\mathrm{u}}}.$$

Show that the equation only depends on the ratio $\alpha = \frac{q_{\mathrm{x}}}{q_{\mathrm{u}}}$ (i.e., rewrite the equation such that α is the only remaining variable)! What is an interpretation of this result? [0.75p]

(e) Assume we use the steady-state Kalman gain

$$K_\infty = \frac{\frac{r_1}{2} + \sqrt{\frac{r_1^2}{4} + r_1 r_2}}{\frac{r_1}{2} + \sqrt{\frac{r_1^2}{4} + r_1 r_2} + r_2}$$

to estimate the state and the LQR gain from Part (d) for state-feedback. Determine the poles of the closed-loop system as a function of r_1, r_2, q_{x}, and q_{u} (if you solved (d), you can use the formulation with α instead of q_{x} and q_{u}). [0.75p]

(f) * Bonus question: Provide an interpretation for how α influences the location of the closed-loop poles. For this, think about what happens for $\alpha \to 0$ and $\alpha \to \infty$ (or $q_{\mathrm{u}} \gg q_{\mathrm{x}}$, $q_{\mathrm{x}} \gg q_{\mathrm{u}}$). [1p*]

(g) Assume we want to add the constraints $|x| < 10$, $|u| < 4$ to the optimization problem. Could we still use an LQG approach as above? If yes, how would we incorporate the constraints (no need to write equations, only outline the idea)? If not, why not and what would be an alternative? [0.5p]

(h) For analyzing stability, we could use Lyapunov's method. Is the existence of a valid Lyapunov function a necessary or sufficient (or both) criterion for closed-loop stability? Does it make a difference whether the system is linear or nonlinear? [0.5p]

Solution. (a) We have a scalar system for which $\Phi = 2$, $\Gamma = C$. Thus, the equation for the Kalman filter reduces to

$$K[k] = \frac{P[k]}{P[k] + r_2}$$

with

$$P[k+1] = P[k] + r_1 - \frac{P[k]^2}{P[k] + r_2}.$$

We now need P_∞, which we get by setting

$$P_\infty = P_\infty + r_1 - \frac{P_\infty^2}{P_\infty + r_2}.$$

Solving for P_∞, we find

$$0 = -P_\infty^2 + r_1 P_\infty + r_1 r_2.$$

We get the two solutions of the quadratic equation through

$$p_{1,2} = \frac{-r_1 \pm \sqrt{r_1^2 + 4 r_1 r_2}}{-2}.$$

Since the covariance must be positive, we know that only the positive solution counts. We can then derive the Kalman gain as

$$\begin{aligned} K_\infty &= \frac{P_\infty}{P_\infty + r_2} \\ &= \frac{\frac{r_1}{2} + \sqrt{\frac{r_1^2}{4} + r_1 r_2}}{\frac{r_1}{2} + \sqrt{\frac{r_1^2}{4} + r_1 r_2} + r_2}. \end{aligned}$$

b) The Kalman filter prediction has zero-mean, i.e., is unbiased, and has minimum variance.

c) From the given equations, we find that

$$L = \frac{S}{S + q_\text{u}},$$

where

$$S = S - \frac{S^2}{S + q_\text{u}} + q_\text{x}.$$

From this, we find

$$0 = -S^2 + S q_\text{x} + q_\text{u} q_\text{x}.$$

This is the exact same structure as in Part a) before, so we can directly find

$$L = \frac{\frac{q_\text{x}}{2} + \sqrt{\frac{q_\text{x}^2}{4} + q_\text{x} q_\text{u}}}{\frac{q_\text{x}}{2} + \sqrt{\frac{q_\text{x}^2}{2} + q_\text{x} q_\text{u}} + q_\text{u}}.$$

d) All we have to do is divide numerator and denominator by q_u:

$$L = \frac{\frac{q_\text{x}}{2} + \sqrt{\frac{q_\text{x}^2}{4} + q_\text{x} q_\text{u}}}{\frac{q_\text{x}}{2} + \sqrt{\frac{q_\text{x}^2}{2} + q_\text{x} q_\text{u}} + q_\text{u}} = \frac{\frac{q_\text{x}}{2q_\text{u}} + \sqrt{\frac{q_\text{x}^2}{4q_\text{u}^2} + \frac{q_\text{x}}{q_\text{u}}}}{\frac{q_\text{x}}{2q_\text{u}} + \sqrt{\frac{q_\text{x}^2}{4q_\text{u}^2} + \frac{q_\text{x}}{q_\text{u}}} + 1} = \frac{\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + \alpha}}{\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + \alpha} + 1}.$$

The result shows that the absolute values of q_u, and q_x are not important for the controller gain, only how large they are relative to each other matters.

e) Due to the separation principle, we can find the poles independent of each other at $\Phi - KC$ and $\Phi - \Gamma L$.
Starting with the observer, we have

$$\Phi - KC = 1 - \frac{\frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} + \beta}}{\frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} + \beta} + 1} = \frac{1}{\frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} + \beta} + 1}$$

and, for the controller,

$$\Phi - \Gamma L = 1 - \frac{\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + \alpha}}{\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + \alpha} + 1} = \frac{1}{\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + \alpha} + 1}.$$

f)* For $\alpha \to 0$, we have poles at 1, i.e., poles that are marginally stable. For $\alpha \to \infty$, the poles are at the origin. If we mainly care about bringing the state to zero (high α), we end up with a deadbeat controller that brings the system to the origin as fast as possible.

g) The LQG cannot handle constraints. Thus, for this task, we need a different controller type, for instance, a model predictive controller.

h) Finding a valid Lyapunov function for a given system is, in general, only a sufficient condition: if we find a valid Lyapunov function, then the system is stable. If we do not find a Lyapunov function, then the system might still be stable, as there are infinitely many potential Lyapunov functions. For linear systems, we can always use the quadratic function. In this case, the criterion is necessary *and* sufficient.