ELEC-E8101 Digital and Optimal Control

Midterm Exam (18.10.2023) – Solution

1. You are working in a company called "Full-service automation house," which does automation projects for the industry. Your client asks for a short-written explanation to the following concepts, which appear in your documents. Write those short descriptions. Please note that the client is somewhat aware of continuous-time control theory and automation but is not familiar with digital control and also not with PID-controllers. The client does not want to read long explanations, so the descriptions should be concise. If you think a diagram is a better way to explain a concept, you can also draw. [3p]

   (a) The windup phenomenona and anti-windup in PID-control. [0.5p]

   (b) The aliasing effect. [0.5p]

   (c) A zero-order hold. [0.25p]

   (d) The different components of a PID-controller and their influence on rise time, overshoot, and steady-state error. [1.25p]

   (e) BIBO stability. [0.25p]

   (f) When is a discrete-time system stable, in comparison to a continuous-time system? [0.25p]

   Solution. (a) The windup phenomena describes that when using an integrator in the controller, the integrator can grow to a large value when the system response is slow due to actuator saturation which causes an overshoot as the sign of the input does not change when we cross the reference value.
   Anti-windup describes methods that counteract the windup problem, for instance, by deceasing the value of the integrator once we reach the input saturation.

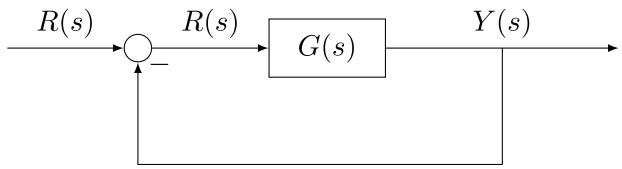
   (b) The aliasing effect occurs when the continuous-time signal is sampled with a frequency lower than the Nyquist frequency, which is two times the natural frequency of the system. Then, we lose information when sampling.

   (c) The signal is kept constant in-between sampling instants.

   (d) The different components of a PID-controller are the proportional gain K_P, the integral gain K_I, and the derivative gain K_D.
   When increasing K_P, the rise time decreases, the overshoot increases, the steady-state error decreases.
   K_P cannot bring the steady-state error to zero.
   When increasing K_I, the rise time decreases, the overshoot increases, and the steady-state error is eliminated.
   When increasing K_D, the rise time does not change much, the overshoot decreases, and the steady-state error is, in theory, not affected.

   (e) A system is called BIBO stable if for a bounded input we receive a bounded output.

(f) While a continuous-time system is stable when the poles of the transfer function have negative real part, a discrete-time system is stable when the poles have an absolute value smaller than 1.



2. Consider the continuous-time system shown in the figure above with transfer function $G(s)$. [4p]

   (a) The continuous-time transfer function of the system is $G(s) = \frac{1}{s^2+s-1}$. Derive the closed-loop transfer function $G_{\mathrm{cl}}(s)$. [0.5p]

   (b) In the lecture, we discussed that we can use the step-invariance method to discretize continuous-time systems. The corresponding formula is [2p]

$$H_{\mathrm{cl}}(z) = \frac{z-1}{z}\mathcal{Z}\left\{ \mathscr{L}^{-1}\left\{\frac{G_{\mathrm{cl}}(s)}{s}\right\}\Bigg|_{t=nT_{\mathrm{s}}} \right\}.$$

   Show that this formula can be expressed using (a selection of) the building blocks given in the table below. You can negate signals by drawing a minus sign on the corresponding arrow. Justify all steps you need to get the equation in a form that lets you represent it using the building blocks.

Impulse	Step	inv. Laplace	z-transf.	Mult.	Sum
		$\mathscr{L}^{-1}$	$\mathcal{Z}$	$\times$	
Derivative	Delay	Prediction	Discr. TF	Cont. TF	Eval.
$\frac{\mathrm{d}}{\mathrm{d}t}$	Delay	Pred.	$H_{\mathrm{cl}}(z)$	$G_{\mathrm{cl}}(s)$	$t = nT_{\mathrm{s}}$

   (c) Assume that $G_{\mathrm{cl}}(s) = \frac{1}{s(s+1)}$. Discretize the system using the backward difference method and $T_{\mathrm{s}} = 1$. [1p]

   (d) What would be an argument *against* using the forward difference method to discretize the system? [0.5p]

Solution. (a) For the closed-loop transfer function, we have

$$\begin{aligned} G_{\mathrm{cl}}(s) &= \frac{G(s)}{1+G(s)} \\ &= \frac{\frac{1}{s^2+s-1}}{1+\frac{1}{s^2+s-1}} \\ &= \frac{1}{s^2+s}. \end{aligned}$$

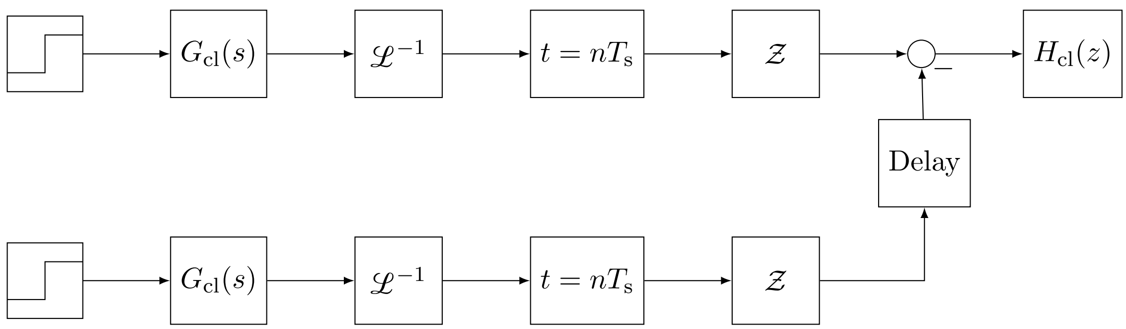
(b) We can first rewrite the fraction as

$$\frac{z-1}{z} = 1 - z^{-1}.$$

We can then rewrite the equation as

$$H_{\mathrm{cl}}(z) = \mathcal{Z}\left\{ \mathscr{L}^{-1}\left\{\frac{G_{\mathrm{cl}}(s)}{s}\right\}\bigg|_{t=nT_{\mathrm{s}}} \right\} - z^{-1}\mathcal{Z}\left\{ \mathscr{L}^{-1}\left\{\frac{G_{\mathrm{cl}}(s)}{s}\right\}\bigg|_{t=nT_{\mathrm{s}}} \right\}.$$

The z^{-1} is a delay, while we can represent the $\frac{1}{s}$ term as a step signal. Hence, we can draw the block diagram:



(c) For the backward difference method, we have $s = \frac{1-z^{-1}}{T_{\mathrm{s}}}$. Inserting this yields

$$\begin{aligned} H_{\mathrm{cl}}(z) &= \frac{1}{(1-z^{-1})(1-z^{-1}+1)} \\ &= \frac{1}{2-3z^{-1}+z^{-2}} \\ &= \frac{z^2}{2z^2-3z+1}. \end{aligned}$$

(d) An argument against using the forward difference method is that it might make a stable continuous-time system unstable.

3. Given a DC motor with the following dynamical equations: [5p]

$$I(t) = \ddot{\theta}(t) + 3\dot{\theta}(t)$$
$$V(t) = \dot{I}(t) + 2I(t).$$

Hint: $x(t) = \begin{pmatrix} \dot{\theta}(t) \\ I(t) \end{pmatrix}$

(a) Derive the state-space representation of this system with $V(t)$ as input and $\dot{\theta}(t)$ as output. [1p]

(b) Assume that the input $V(t)$ is held constant between sampling times, i.e., we use a zero-order hold. Discretize the system with the sampling time $T_\mathrm{s} = 1$.
Hint: $\mathscr{L}^{-1}\left\{\frac{1}{s^2+5s+6}\right\} = e^{-3t}(e^t - 1)$ [2.5p]

(c) Is there an information loss when discretizing the system in that way? [0.5p]

(d) Assume the state matrix is given as

$$\Phi = \begin{pmatrix} 0.045 & 0.086 \\ 0 & 0.135 \end{pmatrix}.$$

Is the system stable? [1p]

Solution. (a) Using the definitions given in the exercise description, we find

$$\dot{x}(t) = \begin{pmatrix} \ddot{\theta}(t) \\ \dot{I}(t) \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 0 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t).$$

(b) Since the system matrix is neither scalar nor sparse, we use the solution approach via the inverse Laplace transform. Then, we have

$$\Phi = e^{AT_\mathrm{s}} = e^A = \mathscr{L}^{-1}\{(sI - A)^{-1}\}.$$

Inserting the A matrix we found above, we get

$$\begin{aligned}
\Phi &= \mathscr{L}^{-1}\left\{\begin{pmatrix} s+3 & -1 \\ 0 & s+2 \end{pmatrix}^{-1}\right\}\Bigg|_{t=1} \\
&= \mathscr{L}^{-1}\left\{\frac{1}{(s+3)(s+2)}\begin{pmatrix} s+2 & 1 \\ 0 & s+3 \end{pmatrix}\right\}\Bigg|_{t=1} \\
&= \mathscr{L}^{-1}\left\{\begin{pmatrix} \frac{1}{s+3} & \frac{1}{s^2+5s+6} \\ 0 & \frac{1}{s+2} \end{pmatrix}\right\}\Bigg|_{t=1} \\
&= \begin{pmatrix} e^{-3} & e^{-3}(e-1) \\ 0 & e^{-2} \end{pmatrix}.
\end{aligned}$$

Next, we derive the Γ matrix:

$$\begin{aligned}\Gamma &= \int_0^1 e^{As}\mathrm{d}sB \\ &= \int_0^1 \begin{pmatrix} e^{-3s} & e^{-3s}(e^s-1) \\ 0 & e^{-2s} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{d}s \\ &= \int_0^1 \begin{pmatrix} e^{-3s}(e^s-1) \\ e^{-2s} \end{pmatrix} \mathrm{d}s \\ &= \begin{pmatrix} \left[-\frac{1}{2}e^{-2s} + \frac{1}{3}e^{-3s}\right]_0^1 \\ \left[-\frac{1}{2}e^{-2s}\right]_0^1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2}e^{-2} + \frac{1}{3}e^{-3} + \frac{1}{6} \\ -\frac{1}{2}e^{-2} + \frac{1}{2} \end{pmatrix}.\end{aligned}$$

Thus, we have the discrete-time state-space representation

$$\begin{aligned} x[k+1] &= \begin{pmatrix} e^{-3} & e^{-3}(e-1) \\ 0 & e^{-2} \end{pmatrix} x[k] + \begin{pmatrix} -\frac{1}{2}e^{-2} + \frac{1}{3}e^{-3} + \frac{1}{6} \\ -\frac{1}{2}e^{-2} + \frac{1}{2} \end{pmatrix} u[k] \\ y[k] &= \begin{pmatrix} 1 & 0 \end{pmatrix} x[k]. \end{aligned}$$

(c) No, this is an exact calculation. As the system is held constant in between sampling times, we always integrate the system state from time t to time $t + T_\mathrm{s}$, as we see in the derivation of Γ.

(d) For the discrete-time system to be stable, we want the eigenvalues of Φ to have absolute value smaller than zero. Thus, we need to solve

$$\begin{aligned}\chi(\lambda) &= \det(\lambda I - \Phi) \\ &= \det \begin{pmatrix} \lambda - 0.045 & -0.086 \\ 0 & \lambda - 0.135 \end{pmatrix} \\ &= (\lambda - 0.045)(\lambda - 0.135) \overset{!}{=} 0\end{aligned}$$

The poles are at $\lambda_1 = 0.045$ and $\lambda_2 = 0.135$, which is both inside the unit circle, hence, the system is stable.