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MS-A0310 / Spring 2017
Final Exam, 24.04.2017

No calculators or notes of any kind are allowed

This exam consists of 6 problems each worth the same number of points.

Question 1: On the attached page you will see the plot of six vector fields labeled (1) - (6).

(a) Define the term *conservative vector field.*

(b) List three conditions which are equivalent to a vector field being conservative. That is, $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ is a vector field if and only if ...... if and only if ...... if and only if .... Mention any extra conditions that are needed for your statements to be true.

(c) Determine which plotted vector fields are conservative. Justify your answer.

(d) Roughly redraw plot (5). Sketch a curve C so that the work done by the vector field along C is negative. Justify your answer.

(e) For plots (5) and (6) write down vector fields that you think made these plots. You don't have to be exactly right to get credit.

Question 2: Let $F = \langle y, x + 1 \rangle$.

(a) Show that F is conservative and then find its potential function $f(x, y)$

(b) Let C be a curve joining the point $(1, 0)$ and $(3, 3)$. Find the work done by F along C.

(c) Find and sketch the equipotential curve (that is, the level curve of f) passing through the point $(1, 1)$

(d) Find the equation for the streamline (flowline) for F passing though the point $(1, 1)$

(e) State a general fact about the intersection of the equipotential curves and streamlines. Show that this fact holds in the equations you found in parts (c) and (d).

Question 3:

Let $\mathbf{F} = \langle xy, y^2 \rangle$. Let γ be the counter-clockwise circle of radius a centered at $(0,0)$.

(a) Compute $\int_\gamma \mathbf{F} \cdot dr$ by first parametrization the curve γ.

(b) Compute $\int_\gamma \mathbf{F} \cdot dr$ by using Green's theorem.

(c) Find a closed curve c so that $\int_c \mathbf{F} \cdot dr \neq 0$. Justify your answer.

(d) Your friend argues that $\mathbf{F}$ is conservative because of the result in part (a). Is your friend correct? Explain what is right/wrong with their reasoning.

Question 4: Refer to the attached plots.

(a) Briefly explain the physical meaning of the curl and divergence of a vector field.

(b) For plot (4), give the location of points A and B so that divergence of the vector field is positive at A and negative at B. Explain.

(c) On plot (5), at the point with coordinates $(x, y) = (7, 0)$, is the curl positive, negative or zero. Explain.

(d) Let $f(x, y, z)$ be a smooth function defined on all of $\mathbb{R}^3$. Show in general that the curl of the gradient of f is zero. This is $\text{curl}(\nabla f) = 0$.

Question 5: Let $\mathbf{F} = \langle y, z + (z-2)^3, x \rangle$. Let the surface S_1 be the graph of $z = \sqrt{x^2 + y^2}$ and S_2 be the graph of $6 - x^2 - y^2$. Let C be the curve of intersection of S_1 and S_2 oriented clockwise when viewed from above.

(a) Sketch the surfaces S_1 and S_2 on the same set of axes.

(b) Find a parametric equation for C.

(c) Compute $\int_C \mathbf{F} \cdot dr$ using part (b).

(d) Compute $\int_C \mathbf{F} \cdot dr$ using Stokes' theorem. Did you get the same answer? *Hint: Choose a surface for which the calculations are the easiest.*

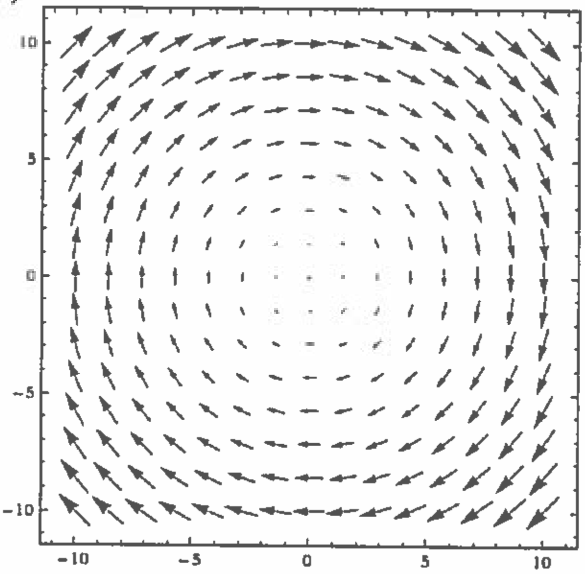
Question 6: Let E be the solid region lying above $z = \sqrt{x^2 + y^2}$ and inside $x^2 + y^2 + z^2 = 1$. Let $F = \langle x, y, z \rangle$.

(a) Find the downward flux over the part of the surface of E formed by $z = \sqrt{x^2 + y^2}$. *Hint: If you wish you can use a sketch to determine this value with no calculation.*

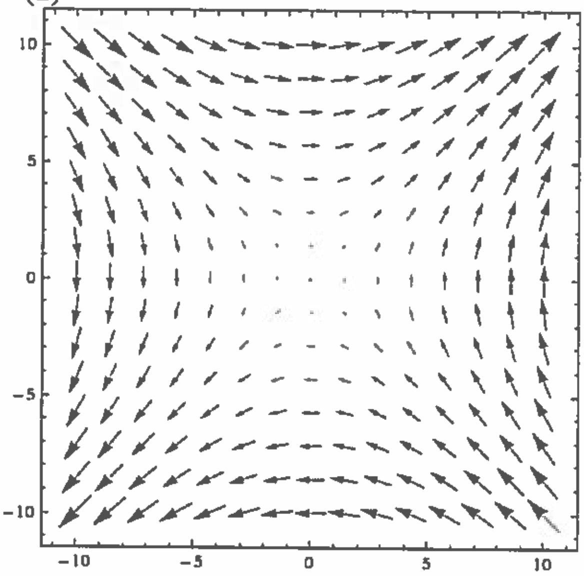
(b) Use the divergence theorem to compute the outward flux of F over the surface of E. (You can solve this even if you have not done part (a)).

Vector field plots

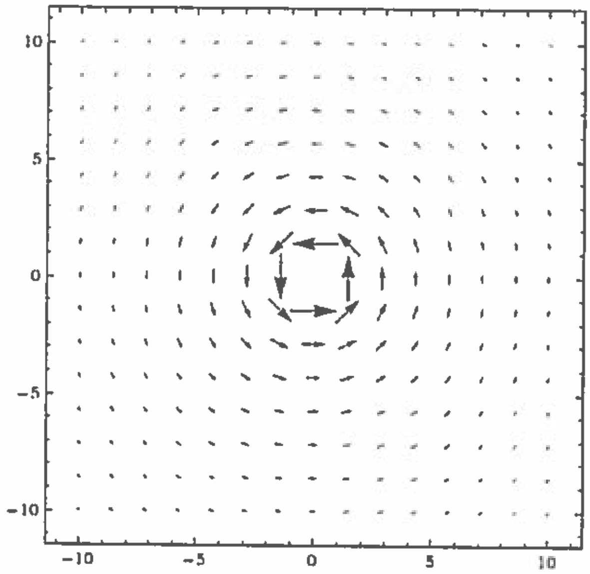
(1)



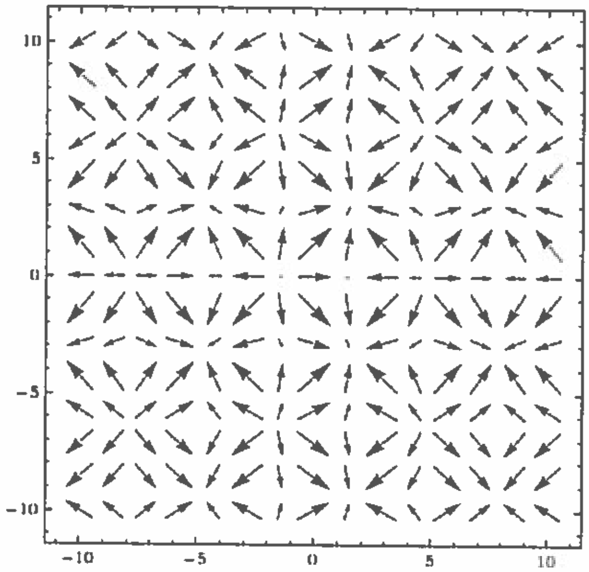
(2)



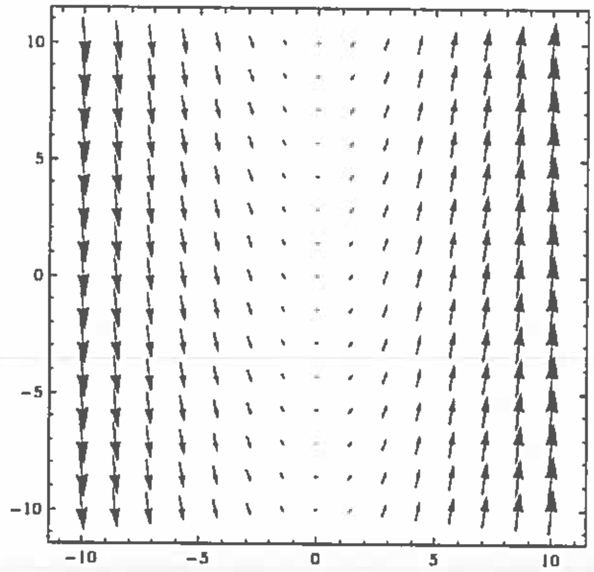
(3)



(4)



(5)



(6)