MS-C1080 EXAM
ALGEBRAN PERUSRAKENTEET
INTRODUCTION TO ABSTRACT ALGEBRA
25.04.2017 (3h)

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In all the assignments, a ring is assumed to have (by definition) an identity element: $1_R \in R$.
You may answer in either Finnish or English. No calculators or tables are allowed.

1. Define/explain the following concepts:
   a) (2p) Group and quotient group.
   b) (2p) Ring and ideal.
   c) (3p) Ring homomorphism. Kernel and image (of a ring homomorphism).

2. (6p) Let $C_\infty = \langle c \rangle$ be an infinite cyclic group. Show that if $n > 0$, then

$$C_\infty / \langle c^n \rangle \simeq C_n,$$

   where C_n is a finite cyclic group with n elements.

3. a) (3p) Define $[G : H]$, the index of a subgroup H of G in G.
   b) (3p) State and prove Lagrange's index theorem.

4. (6p) Prove that a subgroup of a group can be the kernel of a group homomorphism if and only if the subgroup is normal.

5. Let $\mathbb{Z}[i] = \{n + im \mid n, m \in \mathbb{Z}\}$ be the ring of Gaussian integers ($i = \sqrt{-1}$) and let $a - ib \in \mathbb{Z}[i]$ so that $\gcd(a, b) = 1$.
   a) (5p) Show that

$$\mathbb{Z}[i]/\langle a - ib \rangle \simeq \mathbb{Z}/(a^2 + b^2)\mathbb{Z}.$$

   (Hint: Consider the canonical projection $\pi : \mathbb{Z} \to \mathbb{Z}[i]/\langle a - ib \rangle$, $\pi(n) = n + \langle a - ib \rangle$. You can assume that this map is surjective).
   b) (2p) When is the quotient $\mathbb{Z}[i]/\langle a - ib \rangle$ a field?

[Extra] You can earn two bonus points for proving that the canonical projection in part a) is surjective.

Suomenkielinen tentti kääntöpuolella!