Aalto University
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2023-2024/III

**Exam T0 / ET01 / KT**
22.02.2024
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*Allowed equipment:*

- *writing equipment,*
- *memory aid sheet (A4, hand-written, one-sided, w/ name + student number).*

This is the exam sheet for both the final exam (T0/ET01) and the course exam (KT) of MS-C1541 Metric spaces. The grading is based on either

- 100% final exam (T0/ET01);
- 60% course exam (KT) + 40% exercises (during the period III course).

You can attempt both options, and the one leading to the more favorable grade is taken into account.

Depending on the option above, you should solve the following problems:

- **Final exam (T0/ET01):** Solve all five problems.
- **Course exam (KT):** Choose any four of the five problems.

If you solve all problems, the best four are taken into consideration for the course completion option based on course exam + exercises.

## Problems

**Problem 1.**
Possible or not? In each subproblem below, give an example of the specified kind, or explain why such an example cannot exist. In your justifications for the requested properties or their impossibility you can (and should) use results from the course.

(a) A real-number sequence $(x_n)_{n\in\mathbb{N}}$ which has infinitely many zeroes (i.e., $\#\{n \in \mathbb{N} \mid x_n = 0\} = \infty$) but which does not converge to zero. **(3 pts)**

(b) A continuous real-valued function $f: C \to \mathbb{R}$ on the cube

$$C = \big\{(x, y, z) \in \mathbb{R}^3 \;\big|\; 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\big\},$$

such that $f$ has no maximum. **(3 pts)**

**Problem 2.**
Consider the vector space $\mathbb{R}^3 = \big\{(x, y, z) \;\big|\; x, y, z \in \mathbb{R}\big\}$.

For $(x, y, z),\ (x', y', z') \in \mathbb{R}^3$, define

$$\langle (x, y, z),\ (x', y', z') \rangle = xx' - 2xy' - 2yx' + 4yy' + zz' .$$

Among the required properties of an *inner product*, which ones hold and which ones do not hold for $\langle \cdot, \cdot \rangle$? Justify your answers. **(6 pts)**

**Problem 3.**
Let $X = (0, \infty) \subset \mathbb{R}$ be the set of positive real numbers, equipped with the metric inherited from the real line $\mathbb{R}$.

(a) Show that the formula $f(x) = \frac{1}{x}$ defines a homeomorphism $f \colon X \to X$. **(2 pts)**
<u>Hint</u>: *The continuity of rational functions can be considered known.*
(b) Show that the sequence $(x_n)_{n\in\mathbb{N}}$ given by $x_n = \frac{1}{n}$ is Cauchy in $X$. **(2 pts)**
(c) Let $f \colon X \to X$ be the homeomorphism of part (a), and let $(x_n)_{n\in\mathbb{N}}$ be the sequence of part (b). Consider the sequence $(y_n)_{n\in\mathbb{N}}$ given by $y_n = f(x_n)$ for $n \in \mathbb{N}$. Is $(y_n)_{n\in\mathbb{N}}$ Cauchy in $X$? **(2 pts)**

**Problem 4.**
Let $(X, \mathsf{d}_X)$, $(Y, \mathsf{d}_Y)$, $(Z, \mathsf{d}_Z)$ be three metric spaces.

(a) Consider the Cartesian product $X \times Y$, and define

$$\mathsf{d} \colon (X \times Y) \times (X \times Y) \to [0, +\infty)$$

by the formula

$$\mathsf{d}\big((x_1, y_1), (x_2, y_2)\big) = \max\{\mathsf{d}_X(x_1, x_2), \mathsf{d}_Y(y_1, y_2)\}$$

for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$. Check that $\mathsf{d}$ thus defined gives a metric on the product space $X \times Y$. **(3 pts)**
(b) Let $f \colon Z \to X$ and $g \colon Z \to Y$ be two continuous functions. Consider the function

$$h \colon Z \to X \times Y \quad \text{given by} \quad h(z) = \big(f(z), g(z)\big) \text{ for } z \in Z.$$

Prove directly using the $\varepsilon$-$\delta$ definition of continuity that the function $h$ is also continuous, when the Cartesian product space $X \times Y$ is equipped with the metric $\mathsf{d}$ of part (a). **(3 pts)**

**Problem 5.**
Let $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$f\big((x, y)\big) = \Big(1 - \frac{y}{6} + \frac{1}{5}\sin(x - y),\ 2 - \cos\Big(\frac{x + y}{8}\Big)\Big) \qquad \text{for } (x, y) \in \mathbb{R}^2.$$

(a) Show that for any $(x, y), (x', y') \in \mathbb{R}^2$ we have

$$\Big\| f\big((x, y)\big) - f\big((x', y')\big) \Big\| \leq \frac{5}{6}\, \|(x, y) - (x', y')\|,$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^2$. **(2 pts)**
<u>Hint</u>: *The functions* $\sin \colon \mathbb{R} \to \mathbb{R}$ *and* $\cos \colon \mathbb{R} \to \mathbb{R}$ *are 1-Lipschitz (this can be considered known for the purposes of the present problem).*
(b) Define a sequence $(v_n)_{n\in\mathbb{N}}$ in the Euclidean plane $\mathbb{R}^2$ recursively by setting

$$v_0 = (x_0, y_0) := (1, 0) \qquad \text{and}$$
$$v_n = (x_n, y_n) := f\big((x_{n-1}, y_{n-1})\big) \qquad \text{for } n \in \mathbb{N}.$$

Using results from the course, show that the sequence $(v_n)_{n\in\mathbb{N}}$ converges to a limit $\tilde{v} = (\tilde{x}, \tilde{y}) \in \mathbb{R}^2$, and that the coordinates of this limit satisfy the equations $\tilde{x} = \frac{1}{6}\Big(4 + \cos\big(\frac{\tilde{x}+\tilde{y}}{8}\big) + \frac{6}{5}\sin(\tilde{x} - \tilde{y})\Big)$ and $\tilde{y} = 2 - \cos\big(\frac{\tilde{x}+\tilde{y}}{8}\big)$. **(4 pts)**