Abstract Algebra Exam, MS-C1081

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You may bring to the exam a calculator and a memory aid sheet of size A4. The memory aid sheet must be hand-written, contain text on one side only, and have your name and student number written in the top right corner. You do not need to return the memory aid sheet.

- If you are taking MS-C1081 course exam (KT), then do problems 1-4. Your final points for the course will be 10/7*(exam points) + exercise points. You may choose to do the option below, in which case your final points for the course will be maximum of the two options.
- If you are taking MS-C1081 general exam (T0), then do problems 1-5. Your final points for the course will be 50/21*(exam points).

Problems:

1. (a) (5 points) Let G be a group (written additively), S a nonempty set, and $M(S,G)$ the set of all functions $f : S \to G$. Define addition in $M(S,G)$ as follows: for $f, g \in M(S,G)$, the function $f + g$ is given by

$$(f+g)(s) := f(s) + g(s) \text{ for all } s \in S.$$

Prove that $M(S,G)$ is a group. Prove that if G is an abelian group, then $M(S,G)$ is an abelian group.

(b) (4 points) Let R be a ring and S a nonempty set. Prove that the group $M(S,R)$ (see part (a)) is a ring with multiplication defined as follows: for $f, g \in M(S,G)$, the function fg is given by

$$(fg)(s) := f(s)g(s) \text{ for all } s \in S.$$

2. Recall that S_n is the set of all permutations of $\{1, 2, \ldots, n\}$ and it forms a group under taking compositions.

(a) (3 points) Prove that the set $\{\sigma \in S_n : \sigma(n) = n\}$ is a subgroup of S_n for all $n \geq 1$.

(b) (3 points) Prove that the group in part (a) is isomorphic to S_{n-1} for all $n > 1$.

3. (a) (3 points) List all subgroups of the group $(\mathbb{Z}_{12}, +)$. Justify why the list of subgroups is complete.

   (b) (2 points) Determine the order of each element of $(\mathbb{Z}_{12}, +)$ and list the generators of $(\mathbb{Z}_{12}, +)$.

   (c) (4 points) Let G be a group and $\phi : \mathbb{Z}_{12} \to G$ a group homomorphism. What are the possible images $\phi(\mathbb{Z}_{12})$ of $\mathbb{Z}_{12}$ under the group homomorphism ϕ (up to isomorphism)? Justify your answer.

   (d) (3 points) Consider the subgroup $G = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}\}$ of $\mathbb{Z}_{12}$. Give the addition table (Cayley table) for the quotient group $\mathbb{Z}_{12}/G$.

4. (a) (3 points) Let R, S be rings and $f : R \to S$ a ring homomorphism. Let I be an ideal of R. Prove that $f(I)$ is an ideal of $f(R)$.

   (b) (2 points) Give an example of a nonzero ring homomorphism $f : R \to S$ of rings with identity such that $f(1_R) \neq 1_S$.

   (c) (3 points) Let R, S be rings with identity, $f : R \to S$ a ring homomorphism and u a unit in R such that $f(u)$ is a unit in S. Prove that $f(1_R) = 1_S$ and $f(u^{-1}) = f(u)^{-1}$.

5. (a) (4 points) Prove that $\mathbb{R}$ and $\mathbb{C}$ are not isomorphic as rings.

   (b) (3 points) Let R be an integral domain. Show that R is a field if and only if the only ideals in R are $\{0\}$ and R.