Aalto University
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2021/III

Exam KT / T0
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This is the exam sheet for both the course exam (KT) and the final exam (T0) of MS-C1541 Metric spaces. The grading is based on either

- 50% course exam (KT) + 50% exercises (during period III course);
- 100% final exam (T0).

You can attempt both options, and the one leading to the more favorable grade is taken into account.

Depending on the option above, you should solve the following problems:

- **Course exam (KT):** Choose any five of the six problems.
- **Final exam (T0):** Solve all six problems.

(If you solve all problems, the best five are taken into consideration for the course completion option based on course exam + exercises.)

Problems

Problem 1. **(6 pts)**

Consider the following subsets of the Euclidean plane $\mathbb{R}^2$:

$$\begin{aligned} A_1 &= \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; -3 \leq x \leq 3,\ 0 \leq y \leq 8 \right\}; \\ A_2 &= \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; y = x^3 - x \right\}; \\ A_3 &= \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; 5x^4 + 7y^6 < 11 \right\}. \end{aligned}$$

(a) Which of the sets $A_1, A_2, A_3 \subset \mathbb{R}^2$ are open?
(b) Which of the sets $A_1, A_2, A_3 \subset \mathbb{R}^2$ are closed?
(c) Which of the sets $A_1, A_2, A_3 \subset \mathbb{R}^2$ are compact?

(Justify your answers concisely.)

Solution. Here is a table of the yes/no answers — justifications are given afterwards:

	A_1	A_2	A_3
open	N	N	Y
closed	Y	Y	N
compact	Y	N	N

(a): For a set $A \subset X$ to be open *by definition* means that

$$\forall\, z \in A: \quad \exists\, r > 0: \quad \mathcal{B}_r(z) \subset A,$$

where $\mathcal{B}_r(z) = \left\{ w \in X \;\middle|\; \mathsf{d}(z,w) < r \right\}$. The two negative answers in (a) are proven directly from the definition as follows:

- The set A_1 is not open because (for example) for $z = (-3, 8) \in A_1$ no such $r > 0$ exists — indeed for any $r > 0$ we have $(-3,\ 8 + r/2) \in \mathcal{B}_r((-3,8)) \setminus A_1$.
- The set A_2 is not open because (for example) for $z = (0,0) \in A_2$ no such $r > 0$ exists — indeed for any $r > 0$ we have $(0, r/2) \in \mathcal{B}_r((0,0)) \setminus A_2$.

Theorem VI.14 gives a sufficient condition for openness of sets: if $f\colon X \to Y$ is continuous and $V \subset Y$ is open, then the preimage $f^{-1}[V] \subset X$ is open. The positive answer in (a) is proven as follows:

- Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = 5x^4 + 7y^6$. Then f is continuous (polynomials are continuous by Example VIII.14). Now $A_3 = f^{-1}\big[(-\infty, 11)\big]$ is the preimage of the open set $(-\infty, 11) \subset \mathbb{R}$ under f, so $A_3 \subset \mathbb{R}^2$ is open.

(b): Theorem VI.14 gives a sufficient condition for closedness of sets: if $g\colon X \to Y$ is continuous and $F \subset Y$ is open, then the preimage $g^{-1}[F] \subset X$ is open. The positive answers in (b) are proven as follows:

- Let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $g(x,y) = y - x^3 + x$. Then g is continuous (polynomial). We have $A_2 = g^{-1}[\{0\}]$ is the preimage of the singleton $\{0\} \subset \mathbb{R}$ under g. Since singletons are closed sets, $A_2 \subset \mathbb{R}^2$ is closed.
- Let $\mathrm{pr}_1, \mathrm{pr}_2\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $\mathrm{pr}_1(x,y) = x$ and $\mathrm{pr}_2(x,y) = y$. These projections $\mathrm{pr}_1, \mathrm{pr}_2$ are continuous (as polynomials; or more directly as 1-Lipschitz functions). We have $A_1 = \mathrm{pr}_1^{-1}\big[[-3,3]\big] \cap \mathrm{pr}_2^{-1}\big[[0,8]\big]$. The two terms here are preimages of closed intervals $[a,b] \subset \mathbb{R}$ under the (continuous) projections, and therefore closed. Intersections of closed sets are closed by Proposition V.31, so also A_1 is closed.

For a set $A \subset X$ to be closed means, according to the characterization of Proposition VIII.7, that for every sequence $(z_n)_{n\in\mathbb{N}}$ in A which converges in X we have $\lim_{n\to\infty} z_n \in A$. The negative answer in (b) is justified using this characterization as follows:

- The set A_3 is not closed because the sequence $(z_n)_{n\in\mathbb{N}}$ with
$$z_n = \Big(\sqrt[4]{\frac{11}{5}} - \frac{1}{n},\ 0\Big) \in A_3$$
has limit $\lim_{n\to\infty} z_n = (\sqrt[4]{\frac{11}{5}}, 0) \in \mathbb{R}^2 \setminus A_3$.

(c): By the Bolzano-Weierstrass theorem, a subset $A \subset \mathbb{R}^2$ of the Euclidean space $\mathbb{R}^2$ is compact if and only if it is closed and bounded.

- The set A_1 is closed by (b) and bounded since $A_1 \in \overline{\mathcal{B}}_r((0,0))$ with $r = \sqrt{3^2+8^2} = \sqrt{73}$. Therefore A_1 is compact.
- The set A_2 is not bounded: for any $r > 0$ we have $z = (r, r^3 - r) \in A_2$ with $\mathsf{d}\big((r, r^3 - r),\ (0,0)\big) > r$. Therefore A_2 is not compact.
- The set A_3 is not closed by (b). Therefore A_2 is not compact.

There are very many different ways to justify the correct statements (using the definitions directly, or any of the various characterizations). They differ significantly in the details. Points are given by carefully justified correct answers, based on clearly stated results from the course. Partial credit is also given for justifications using clearly stated correct results not contained in the course.

No points are given for unclearly stated reasoning — even of correct statements (guessing yes/no correctly is no evidence of understanding!). However, partial credit may be given for incorrect conclusions obtained by correct and relevant reasoning from a mistaken premise.

Problem 2. **(6 pts)**

Let (X, d_X) and (Y, d_Y) be two metric spaces. Suppose that $f\colon X \to Y$ is a function such that for some $\alpha > 0$ and $M > 0$ we have

$$\mathsf{d}_Y\big(f(x), f(x')\big) \;\leq\; M\,\big(\mathsf{d}_X(x, x')\big)^{\alpha} \qquad \text{for all } x, x' \in X.$$

Prove that if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X, then $\big(f(x_n)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y.

Solution. Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy w.r.t. metric d if

$$\forall\, \varepsilon > 0: \quad \exists\, n_\varepsilon \in \mathbb{N}: \quad \forall\, k, \ell \geq n_\varepsilon: \quad \mathsf{d}(x_k, x_\ell) < \varepsilon.$$

Assume that $(x_n)_{n\in\mathbb{N}}$ in Cauchy in X w.r.t. d_X, and let $\alpha > 0$ and $M > 0$ be constants for which the given property of f holds.

We will show that $\big(f(x_n)\big)_{n\in\mathbb{N}}$ is Cauchy in Y w.r.t. d_Y. So let $\varepsilon > 0$. Then since $(\varepsilon/M)^{1/\alpha} > 0$, by the Cauchy property of $(x_n)_{n\in\mathbb{N}}$ there exists a $n' \in \mathbb{N}$ such that for all $k, \ell \geq n'$ we have

$$\mathsf{d}_X(x_k, x_\ell) < (\varepsilon/M)^{1/\alpha}.$$

Using the assumption on f, we get for all $k, \ell \geq n'$ that

$$\begin{aligned} \mathsf{d}_Y\big(f(x_k), f(x_\ell)\big) \;&\leq\; M\,\big(\mathsf{d}_X(x_k, x_\ell)\big)^{\alpha} \\ &<\; M\,\big((\varepsilon/M)^{1/\alpha}\big)^{\alpha} \\ &=\; M\,(\varepsilon/M) \;= \varepsilon. \end{aligned}$$

This proves that $\big(f(x_n)\big)_{n\in\mathbb{N}}$ is Cauchy.

(1) The correct definition of a Cauchy sequence.
(Order of quantifiers!)
(2) Cauchy properties considered w.r.t. the right metrics.
(3) Mention the relevant assumptions on f and $(x_n)_{n\in\mathbb{N}}$ as they are used.
(In particular, fix the constants M, α!)
(4) Select n' suitably using the assumed Cauchy property.
(5) Calculate to show $\mathsf{d}_Y\big(f(x_k), f(x_\ell)\big) < \varepsilon$ for $k, \ell \geq n'$.
(6) Concluding correctly.

Problem 3. **(6 pts)**

Let (X, d_X) and (Y, d_Y) be two metric spaces. Consider the Cartesian product

$$X \times Y = \Big\{ (x,y) \;\Big|\; x \in X,\ y \in Y \Big\},$$

consisting of ordered pairs (x, y) with the first component x in X and second component y in Y.

Show directly from the definitions that the formula

$$\mathsf{d}\big((x_1,y_1),(x_2,y_2)\big) = \mathsf{d}_X(x_1,x_2) + \mathsf{d}_Y(y_1,y_2) \quad \text{for } (x_1,y_1),(x_2,y_2) \in X \times Y$$

defines a metric on $X \times Y$.

Solution. To prove that the given formula defines a metric d on the Cartesian product $X \times Y$, according to Definition V.1 we must show that it defines a function

$$\mathsf{d} \colon (X \times Y) \times (X \times Y) \to [0, \infty)$$

which satisfies properties "symmetricity" **(M-s)**, "triangle inequality" **(M-Δ)**, and "separation" **(M-0)**. The given formula

$$\mathsf{d}\big((x_1,y_1),(x_2,y_2)\big) = \mathsf{d}_X(x_1,x_2) + \mathsf{d}_Y(y_1,y_2) \quad \text{for } (x_1,y_1),(x_2,y_2) \in X \times Y$$

defines a function on the domain $(X \times Y) \times (X \times Y)$. The codomain can be taken as $[0, \infty)$, since the metrics d_X and d_Y are non-negative, and we therefore obtain the non-negativity of d:

$$\mathsf{d}\big((x_1,y_1),(x_2,y_2)\big) = \underbrace{\mathsf{d}_X(x_1,x_2)}_{\geq 0} + \underbrace{\mathsf{d}_Y(y_1,y_2)}_{\geq 0} \geq 0.$$

Let us then check each of the the three remaining conditions separately.

(M-s): We must prove

$$\forall\, (x_1,y_1),(x_2,y_2) \in X \times Y : \quad \mathsf{d}\big((x_1,y_1),(x_2,y_2)\big) = \mathsf{d}\big((x_2,y_2),(x_1,y_1)\big).$$

Let $(x_1,y_1),(x_2,y_2) \in X \times Y$. Then by the property **(M-s)** of the metrics d_X and d_Y and the definition of d, we have

$$\mathsf{d}\big((x_1,y_1),(x_2,y_2)\big) = \underbrace{\mathsf{d}_X(x_1,x_2)}_{=\mathsf{d}_X(x_2,x_1)} + \underbrace{\mathsf{d}_Y(y_1,y_2)}_{=\mathsf{d}_Y(y_2,y_1)} = \mathsf{d}\big((x_2,y_2),(x_1,y_1)\big).$$

This shows **(M-s)** for d.

(M-0): We must prove

$$\forall\, (x_1,y_1),(x_2,y_2) \in X \times Y : \quad \mathsf{d}\big((x_1,y_1),(x_2,y_2)\big) = 0 \;\Leftrightarrow\; (x_1,y_1) = (x_2,y_2).$$

Let $(x_1,y_1),(x_2,y_2) \in X \times Y$. We prove the two implications separately.

Assume first that $(x_1,y_1) = (x_2,y_2)$. By property **(M-0)** of the metrics d_X and d_Y, we then get

$$\mathsf{d}\big((x_1,y_1),(x_1,y_1)\big) = \underbrace{\mathsf{d}_X(x_1,x_1)}_{=0} + \underbrace{\mathsf{d}_Y(y_1,y_1)}_{=0} = 0 + 0 = 0.$$

This proves the "$\Leftarrow$" implication.

Assume then that $\mathsf{d}\big((x_1,y_1),(x_2,y_2)\big) = 0$. By non-negativity of the two terms in the definition of d, this is only possible if $\mathsf{d}_X(x_1,x_2) = 0$ and $\mathsf{d}_Y(y_1,y_2) = 0$. By property **(M-0)** of the metrics d_X and d_Y, these are only possible if $x_1 = x_2$ and $y_1 = y_2$. We get $(x_1,y_1) = (x_2,y_2)$. This proves the "$\Rightarrow$" implication.

(M-Δ): We must prove

$$\forall\ (x_1,y_1),(x_2,y_2),(x_3,y_3) \in X \times Y :$$
$$\mathsf{d}\big((x_1,y_1),(x_3,y_3)\big) \le \mathsf{d}\big((x_1,y_1),(x_2,y_2)\big) + \mathsf{d}\big((x_2,y_2),(x_3,y_3)\big).$$

Let $(x_1,y_1),(x_2,y_2),(x_3,y_3) \in X\times Y$. Then by the property **(M-Δ)** of the metrics d_X and d_Y and the definition of d, we have

$$\begin{aligned}
\mathsf{d}\big((x_1,y_1),(x_3,y_3)\big) &= \underbrace{\mathsf{d}_X(x_1,x_3)}_{\le \mathsf{d}_X(x_1,x_2)+\mathsf{d}_X(x_2,x_3)} + \underbrace{\mathsf{d}_Y(y_1,y_3)}_{\le \mathsf{d}_Y(y_1,y_2)+\mathsf{d}_Y(y_2,y_3)} \\
&\le \mathsf{d}_X(x_1,x_2) + \mathsf{d}_X(x_2,x_3) + \mathsf{d}_Y(y_1,y_2) + \mathsf{d}_Y(y_2,y_3) \\
&= \mathsf{d}_X(x_1,x_2) + \mathsf{d}_Y(y_1,y_2) + \mathsf{d}_X(x_2,x_3) + \mathsf{d}_Y(y_2,y_3) \\
&= \mathsf{d}\big((x_1,y_1),(x_2,y_2)\big) + \mathsf{d}\big((x_2,y_2),(x_3,y_3)\big).
\end{aligned}$$

This shows **(M-Δ)** for d.

(1) Domain and codomain of a metric; $\mathsf{d}\colon Z\times Z \to [0,\infty)$, with $Z = X\times Y$ here, and non-negativity of values (codomain)
(2) **(M-s)**, statement and justification
(3) **(M-0)**, one implication (and correct statement)
(4) **(M-0)**, the other implication (and correct statement)
(5) **(M-Δ)**, started meaningfully by fixing three arbitrary points in $X\times Y$
(6) **(M-Δ)**, calculations carried out correctly to conclude the triangle inequality

Problem 4. **(6 pts)**

(a) Let X be a compact metric space and $f\colon X \to \mathbb{R}$ a continuous function such that $f(x) > 0$ for all $x \in X$. Prove (using results from the course) that there exists a $c > 0$ such that $f(x) \geq c$ for all $x \in X$.

(b) Give an example of a continuous function $g\colon Y \to \mathbb{R}$ on a non-compact metric space Y such that $g(x) > 0$ for all $x \in Y$ but there does not exist any $c > 0$ such that $g(x) \geq c$ for all $x \in Y$.
(Detailed proofs of continuity and non-compactness are not required here.)

(c) Give an example of a non-continuous function $h\colon Z \to \mathbb{R}$ on a compact metric space Z such that $h(x) > 0$ for all $x \in Z$ but there does not exist any $c > 0$ such that $h(x) \geq c$ for all $x \in Z$.
(Detailed proofs of non-continuity and compactness are not required here.)

Solution.

(a): The assertion holds trivially if $X = \emptyset$, so assume $X \neq \emptyset$. By Theorem XI.13, a continuous function on a non-empty compact set attains its minimum, i.e., there exists a point $x_0 \in X$ such that $f(x_0) \leq f(x)$ for all $x \in X$. By the positivity assumption $f(x_0) > 0$. This minimum value $c = f(x_0) > 0$ serves as a positive lower bound for the values of f.

(b): Let $Y = (0, \infty)$ and define $g\colon Y \to \mathbb{R}$ by the formula

$$g(y) = y \qquad \text{for } y \in (0, \infty) = Y.$$

The open interval $Y = (0, \infty)$ is not compact (it is neither bounded nor closed, both of which would be required for compactness by the Bolzano-Weierstrass theorem in $\mathbb{R}$). The function g is positive: $f(y) = y > 0$ for every $y \in (0, \infty)$. The greatest lower bound for the set of its values is zero, $\inf \big\{ g(y) \;\big|\; y \in (0, \infty) \big\} = \inf \big\{ y \;\big|\; y \in (0, \infty) \big\} = 0$, so no positive $c > 0$ can be a lower bound. (A detailed proof: for every $c > 0$, at $y = c/2 \in (0, \infty)$ we have $f(y) = y = c/2 < c$, so c is not a lower bound).

(c): Let $Z = [0, 1]$ and define $h\colon Z \to \mathbb{R}$ by the formula

$$h(z) = \begin{cases} 42 & \text{for } z = 0 \\ z & \text{for } z \in (0, 1]. \end{cases}$$

The closed interval $Z = [0, 1]$ is compact (Example XI.2). The function h is positive: for $z = 0$ we have $h(0) = 42 > 0$ and for $z \in (0, 1]$ we have $h(z) = z > 0$. The greatest lower bound for the set of its values is zero, $\inf \big\{ h(z) \;\big|\; z \in [0, 1] \big\} = \inf \Big(\big\{ z \;\big|\; z \in (0, 1] \big\} \cup \{42\} \Big) = 0$, so no positive $c > 0$ can be a lower bound. (A detailed proof: for every $c > 0$, at $z = \min\{c/2, 1\} \in (0, 1]$ we have $h(z) = z \leq c/2 < c$, so c is not a lower bound).

(1) (a) Compactness+continuity imply existence of minimum.
(2) (a) Positivity of the minimum value follows from the assumption, and the minimum value is a lower bound.
(3) (b) An example which works: continuous positive g on a non-compact Y which has no positive lower bound.
(4) (b) Justifications, in particular of positivity but no positive lower bound.
(5) (c) An example which works: non-continuous positive g on a compact Z which has no positive lower bound.
(6) (c) Justifications, in particular of positivity but no positive lower bound.

Problem 5. **(6 pts)**

Let (X, d_X) and (Y, d_Y) be two metric spaces and $f\colon X \to Y$ a continuous function. Suppose that X is path-connected. Show that then the image $f[X] \subset Y$ is also path-connected.

Solution. The assumption that X is path-connected means that for all $x, x' \in X$ there exists a continuous $\gamma\colon [a,b] \to X$ such that $\gamma(a) = x$, $\gamma(b) = x'$.

To prove the path-connectedness of $f[X] \subset Y$, let $y, y' \in f[X]$ be arbitrary. By definition of the image $f[X]$, there then exists some $x, x' \in X$ such that $f(x) = y$ and $f(x') = y'$. By path-connectedness of X, we can choose a continuous $\gamma\colon [a,b] \to X$ such that $\gamma(a) = x$, $\gamma(b) = x'$. Now consider the composition

$$f \circ \gamma\colon [a,b] \to f[X] \subset Y;$$

where we noted that the codomain can be restricted to the range $f[X] \subset Y$ of f. As a composition of continuous functions, this is continuous, so a path in $f[X]$. Its endpoints are

$$(f \circ \gamma)(a) = f(\gamma(a)) = f(x) = y \qquad \text{and} \qquad (f \circ \gamma)(b) = f(\gamma(b)) = f(x') = y'.$$

The existence of such a path for arbitrary y, y' in the image shows the path-connectedness of the image.

(1) Definition of path-connectedness. (Order of quantifiers!)
(2) Definition of image $f[X]$.
(3) Picked arbitrary $y, y' \in f[X]$ and chose a path γ between preimages in X.
(4) Constructed the path $f \circ \gamma\colon [a,b] \to Y$.
(5) Checked that $f \circ \gamma$ is continuous (composition) and has the right endpoints.
(6) Concluding the path-connectedness of the image $f[X]$
(Other details relevant to the conclusion: values of $f \circ \gamma$ in the image $f[X]$, existence but no claims of uniqueness, ...)

Problem 6. **(6 pts)**

(a) Prove (using results from the course and exercises) that the series

$$f(x) \;=\; \sum_{n=1}^{\infty} 3^{-n} \, \sin(4^n x)$$

converges for all $x \in \mathbb{R}$ and defines a continuous function $f\colon \mathbb{R} \to \mathbb{R}$.

(b) Prove that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined in part (a) is not differentiable at 0, i.e., the derivative $f'(0)$ does not exist.

<u>Hint</u>: *First calculate the value $f(0)$. It may be useful to show that at the points $x_j = \pi \, 4^{-j}$ for $j \in \mathbb{N}$, one has $f(x_j) \geq \frac{1}{\sqrt{2}} 3^{-j-1}$. To prove non-existence of the derivative $f'(0)$, consider the appropriate difference quotients.*

Solution.

(a): We can apply Weierstrass M-test (Exercise IX.5 in the notes and Exercise 5/5 of the course).

For $n \in \mathbb{N}$, let $M_n = 3^{-n}$ and let $g_n\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$g_n(x) = 3^{-n} \, \sin(4^n x) \qquad \text{for } x \in \mathbb{R}.$$

This function is continuous, as the composition of a constant multiplication $x \mapsto 4^n x$, the 1-Lipschitz function $y \mapsto \sin(y)$, and a constant multiplication $z \mapsto 3^{-n} z$; each seen as functions $\mathbb{R} \to \mathbb{R}$. For $n \in \mathbb{N}$ and $x \in \mathbb{R}$ we have

$$\big|g_n(x)\big| \;=\; \Big|3^{-n} \, \sin(4^n x)\Big| \;=\; 3^{-n} \, \underbrace{\big|\sin(4^n x)\big|}_{\leq 1} \;\leq\; 3^{-n} \;=\; M_n.$$

The geometric series

$$\sum_{n=1}^{\infty} M_n \;=\; \sum_{n=1}^{\infty} 3^{-n} \;=\; \frac{1}{3} \, \frac{1}{1 - 1/3} \;=\; \frac{1}{2}$$

is convergent, because the ratio of its successive terms has absolute value $|\frac{1}{3}| < 1$.

The assumptions of the Weierstrass' M-test

- the continuity of each g_n;
- the bound $|g_n(x)| \leq M_n$ for each n and all $x \in \mathbb{R}$;
- the convergence $\sum_{n=1}^{\infty} M_n < \infty$;

are valid. The conclusion is that the series

$$f(x) \;=\; \sum_{n=1}^{\infty} g_n(x) \;=\; \sum_{n=1}^{\infty} 3^{-n} \, \sin(4^n x)$$

is (uniformly) convergent and defines a continuous function of $x \in \mathbb{R}$.

(b): First calculate the value

$$f(0) \;=\; \sum_{n=1}^{\infty} 3^{-n} \, \underbrace{\sin(4^n 0)}_{=\sin(0)=0} \;=\; \sum_{n=1}^{\infty} 0 \;=\; 0.$$

Now let $x_j = \pi 4^{-j}$ for $j \in \mathbb{N}$. Then consider the value

$$f(x_j) \;=\; \sum_{n=1}^{\infty} 3^{-n} \, \sin(4^n x_j). \;=\; \sum_{n=1}^{\infty} 3^{-n} \, \sin(4^{n-j} \pi).$$

Note that for $n \geq j$ we have that $4^{n-j}\pi$ is an integer multiple of π, and therefore $\sin(4^{n-j}\pi) = 0$. For $n < j$ we have $4^{n-j}\pi \in (0, \pi)$, so $\sin(4^{n-j}\pi) > 0$. We can therefore estimate the sum from below by keeping only the term corresponding to $n = j - 1$:

$$f(x_j) = \sum_{n=1}^{\infty} 3^{-n} \sin(4^{n-j}\pi) = \sum_{n=1}^{j-1} 3^{-n} \sin(4^{n-j}\pi) \geq 3^{-(j-1)} \sin(4^{-1}\pi).$$

Noting that $\sin(\pi/4) = 1/\sqrt{2}$ we get $f(x_j) \geq 3^{1-j}/\sqrt{2}$.

To show that $f'(0)$ does not exist, we argue by contradiction. If

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h - 0} = \lim_{h \to 0} \frac{f(h)}{h}$$

would exist, then in particular along the sequence $(x_j)_{j \in \mathbb{N}}$, which has $\lim_{j\to\infty} x_j = 0$ (and $x_j \neq 0$ for all j), we would get

$$f'(0) = \lim_{j\to\infty} \frac{f(x_j)}{x_j} \geq \lim_{j\to\infty} \frac{3^{1-j}/\sqrt{2}}{\pi 4^{-j}} = \frac{3}{\pi\sqrt{2}} \lim_{j\to\infty} \left(\frac{4}{3}\right)^j = +\infty.$$

This contradiction proves the non-existence of the derivative $f'(0)$.

(1) (a) Convergence at all $x \in \mathbb{R}$, e.g., by Weierstrass' test.
(2) (a) Continuity either by Weierstrass' test or by uniform limit of continuous functions.
(3) (a) Details correct: continuity of the terms mentioned, convergence addressed appropriately etc.
(4) (b) Definition of the derivative as a limit of difference quotients.
(5) (b) Calculations of $f(0)$ and $f(x_j) \geq \cdots$.
(6) (b) From the estimates, concluded the non-existence of the derivative.