Aalto University
Department of Mathematics and Systems Analysis
MS-C1541 — Metric spaces, 2021/III

Exam T01 / KT
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This is the exam sheet for both the the final exam (T01) and for the retake of the course exam (KT) of MS-C1541 Metric spaces. The grading is based on either

- 100% final exam (T01);
- 50% course exam (KT) + 50% exercises (during the period III course).

You can attempt both options, and the one leading to the more favorable grade is taken into account.

Depending on the option above, you should solve the following problems:

- **Final exam (T01):** Solve all six problems.
- **Course exam (KT):** Choose any five of the six problems.

(If you solve all problems, the best five are taken into consideration for the course completion option based on course exam + exercises.)

PROBLEMS

Problem 1. **(6 pts)**

Consider the functions

$$\begin{aligned} d_1\colon\ \mathbb{R}\times\mathbb{R}\to[0,\infty) \qquad & d_1(x,y)=\sqrt{|x-y|} \quad \text{for } x,y\in\mathbb{R},\\ d_2\colon\ \mathbb{R}\times\mathbb{R}\to[0,\infty) \qquad & d_2(x,y)=|x-y|^2 \quad \text{for } x,y\in\mathbb{R}. \end{aligned}$$

Which one of these is a metric on the set $\mathbb{R}$ of real numbers? Prove all conditions of a metric for it. For the other one, show concretely that some required property of a metric fails.

Solution. Recall that a metric on a set X is a function $\mathsf{d}\colon X\times X\to[0,\infty)$ satisfying

$$\begin{array}{lll} \text{(M-s)} & \forall x,y\in X: & \mathsf{d}(x,y) = \mathsf{d}(y,x),\\ \text{(M-}\Delta\text{)} & \forall x,y,z\in X: & \mathsf{d}(x,z) \le \mathsf{d}(x,y)+\mathsf{d}(y,z),\\ \text{(M-0)} & \forall x,y\in X: & \mathsf{d}(x,y)=0 \text{ if and only if } x=y. \end{array}$$

In this problem, we consider metrics on the set $X=\mathbb{R}$ of real numbers.

We claim that d_1 above is a metric on $\mathbb{R}$. Since it is a function with the right domain and codomain, $d_1\colon\mathbb{R}\times\mathbb{R}\to[0,\infty)$, it suffices to verify the conditions (M-s), (M-Δ), and (M-0).

(M-s): Let $x,y\in\mathbb{R}$. We then have $|x-y|=|y-x|$, so also

$$d_1(x,y) \;=\; \sqrt{|x-y|} \;=\; \sqrt{|y-x|} \;=\; d_1(y,x).$$

(M-Δ): Let $x,y,z\in\mathbb{R}$. Calculate first the square of both sides of the inequality,

$$d_1(x,z)^2 \;=\; \Big(\sqrt{|x-z|}\Big)^2 \;=\; |x-z|,$$

and

$$\begin{aligned} \Big(d_1(x,y)+d_1(y,z)\Big)^2 \;&=\; \Big(\sqrt{|x-y|}+\sqrt{|y-z|}\Big)^2\\ &=\; \Big(\sqrt{|x-y|}\Big)^2+2\sqrt{|x-y|}\,\sqrt{|y-z|}+\Big(\sqrt{|y-z|}\Big)^2\\ &=\; |x-y|+2\sqrt{|x-y|}\,\sqrt{|y-z|}+|y-z|. \end{aligned}$$

Leaving out the non-negative term $2\sqrt{|x-y|}\,\sqrt{|y-z|}$ from the last expression can only make it smaller, so we have $\big(d_1(x,y)+d_1(y,z)\big)^2\ge|x-y|+|y-z|$. By the triangle inequality of the absolute value we have $|x-z|\le|x-y|+|y-z|$, so we conclude $\big(d_1(x,y)+d_1(y,z)\big)^2\ge|x-z|=d_1(x,z)^2$. Noting that the expressions inside the squares here are non-negative, we conclude that taking the (non-negative) square roots preserves the inequalities (since $a\mapsto\sqrt{a}$ is increasing $[0,\infty)\to[0,\infty)$). We thus get the desired conclusion of (M-Δ)

$$d_1(x,y)+d_1(y,z) \;\ge\; d_1(x,z).$$

(M-0): For the "if" direction, note that if $x=y$, then $d_1(x,y)=d_1(x,x)=\sqrt{|x-x|}=\sqrt{0}=0$. For the "only if" direction, start by noting that if $d_1(x,y)=0$, then by definition of d_1 we have $\sqrt{|x-y|}=0$. The square root is zero only if its argument is zero, so this gives $|x-y|=0$. Also the absolute value is zero only if its argument is zero, so this gives $x-y=0$. From here we solve $x=y$.

This finishes the proof that d_1 is a metric on $\mathbb{R}$.

By contrast, d_2 is not a metric on $\mathbb{R}$. To see that, we will check that the property (M-Δ) fails. For a counterexample consider, e.g., $x = 5$, $y = 6$, $z = 7$. Then we can calculate

$$\begin{aligned} d_2(x, y) &= |5 - 6|^2 = 1^2 = 1, \\ d_2(x, z) &= |5 - 7|^2 = 2^2 = 4, \\ d_2(y, z) &= |6 - 7|^2 = 1^2 = 1, \end{aligned}$$

so in violation of (M-Δ), we have

$$d_2(x, z) \;=\; 4 \;>\; 2 \;=\; 1 + 1 \;=\; d_2(x, y) + d_2(y, z).$$

(1) The correct definition of a metric.
(2) Verification of (M-s) for d_1.
(3) Verification of (M-Δ) for d_1.
(4) Verification of (M-0) for d_1.
(5) Correctly stated that d_2 is not a metric, because it fails (M-Δ).
(6) Explicit (counter)example demonstrating the failure of (M-Δ) for d_2.

Problem 2. **(6 pts)**

Suppose that X and Y are metric spaces and $f\colon X \to Y$ is a continuous function.

(a) Is it true in general that if $A \subset X$ is closed, then also

$$f[A] = \Big\{ y \in Y \;\Big|\; y = f(a) \text{ for some } a \in A \Big\} \subset Y$$

is closed? If yes, justify why; if no, give a counterexample.

(b) Is it true in general that if $B \subset Y$ is closed, then also

$$f^{-1}[B] = \Big\{ x \in X \;\Big|\; f(x) \in B \Big\} \subset X$$

is closed? If yes, justify why; if no, give a counterexample.

Solution. **(a):** The image $f[A]$ of a closed set $A \subset X$ under a continuous function $f\colon X \to Y$ is not necessarily closed in Y.

For a concrete counterexample[1], consider for example $X = (0, \infty)$ and $Y = \mathbb{R}$ and $f\colon (0, \infty) \to \mathbb{R}$ given by

$$f(x) = \frac{15}{1+x} \qquad \text{for } x \in (0, \infty),$$

which is continuous, as a rational function. Let $A = [2, \infty) \subset (0, \infty) = X$. Then A is closed: its complement $(0, \infty) \setminus [2, \infty) = (0, 2) = \mathcal{B}_1(1)$ an open ball, and thus open. The image is

$$f[A] \;=\; \left\{ y \in \mathbb{R} \;\middle|\; y = \frac{15}{1+x} \text{ for some } x \geq 2 \right\} \;=\; (0, 5],$$

since for any $x \geq 2$ we have $0 < \frac{15}{1+x} \leq 5$, and for any $y \in (0, 5]$ we have $y = f(x)$ at $x = \frac{15}{y} - 1 \geq 2$. The image $f[A] = (0, 5] \subset \mathbb{R}$ is not closed, since its complement $\mathbb{R} \setminus (0, 5] = (-\infty, 0] \cup (5, \infty)$ is not open: no open ball of positive radius centered at the point $0 \in \mathbb{R} \setminus (0, 5]$ is contained in $\mathbb{R} \setminus (0, 5]$.

(b): It is in general true that for $f\colon X \to Y$ continuous and $B \subset Y$ closed, the preimage $f^{-1}[B] \subset X$ is closed. Indeed, in the course a characterization for continuity was given based on preimages of closed sets: a function f is continuous if and only if for every closed $B \subset Y$ we have that the preimage $f^{-1}[B] \subset X$ is closed. (A similar characterization with preimages of open sets was used even more often, and the two are easily seen equivalent by considering the complemenary subsets). The "only if" implication in this characterization answers the problem question in the affirmative.

(1) Clearly and correctly stated that the claim (a) is not generally valid.
(2) A counterexample, with function, domain, codomain, and subset specified.
(3) Justifications of continuity of f, closedness of A, and non-closedness of $f[A]$.
(4) Clearly and correctly stated that the claim (b) is generally valid.
(5) Correct reference (or proof) to a result which implies (b).
(6) Correct reference (or proof) to a result which implies (b).

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[1] *Remark:* Even easier counterexamples could be constructed as follows. Choose some metric space Y and a subset $X \subset Y$ which is not closed (for example $Y = \mathbb{R}$ and $X = \mathbb{Q} \subset \mathbb{R}$.) . Then X is also a metric space with the induced metric. Define the embedding $f\colon X \to Y$ by $x \mapsto x \in Y$ for $x \in X$. This embedding is clearly continuous (indeed 1-Lipschitz). Setting $A = X$, we have that the whole space $A \subset X$ is closed. The image is clearly $f[A] = f[X] = X \subset Y$, which was not closed.

Problem 3. **(6 pts)**
Suppose that $u, v \in \mathsf{V}$ are vectors in an inner product space $(\mathsf{V}, \langle \cdot, \cdot \rangle)$ such that the inner products among them are

$$\langle u, u \rangle = \frac{5}{9}, \qquad \langle u, v \rangle = -\frac{10}{21}, \qquad \langle v, v \rangle = \frac{20}{49}.$$

Directly using the defining properties of inner products, show that there exists a constant $\alpha \in \mathbb{R}$ such that

$$u = \alpha\, v,$$

and find the value of α.

Hint: *To get started, consider whether $u - \alpha\, v$ could be the zero vector.*

Solution. By definition, an inner product on a vector space V is a function $\langle \cdot, \cdot \rangle \colon \mathsf{V} \times \mathsf{V} \to \mathbb{R}$ which satisfies the following 5 properties

(IP1) $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$ for all $\vec{x}, \vec{y} \in \mathsf{V}$;
(IP2) $\langle c\,\vec{x}, \vec{y} \rangle = c\,\langle \vec{x}, \vec{y} \rangle$ for all $c \in \mathbb{R}$ and $\vec{x}, \vec{y} \in \mathsf{V}$;
(IP3) $\langle \vec{x} + \vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$ for all $\vec{x}, \vec{y}, \vec{z} \in \mathsf{V}$;
(IP4) $\langle \vec{x}, \vec{x} \rangle \geq 0$ for all $\vec{x} \in \mathsf{V}$;
(IP5) $\langle \vec{x}, \vec{x} \rangle = 0$ only if $\vec{x} = \vec{0} \in \mathsf{V}$.

From the first three properties one obtains bilinearity:

$$\begin{aligned} & \langle \alpha_1 x_1 + \alpha_2 x_2,\ \beta_1 y_1 + \beta_2 y_2 \rangle \\ = \ & \alpha_1 \beta_1 \langle x_1, y_1 \rangle + \alpha_1 \beta_2 \langle x_1, y_2 \rangle + \alpha_2 \beta_1 \langle x_2, y_1 \rangle + \alpha_2 \beta_2 \langle x_2, y_2 \rangle \end{aligned}$$

for all $x_1, x_2, y_1, y_2 \in \mathsf{V}$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$.

Assume now $u, v \in \mathsf{V}$ satisfy the conditions in the problem statement, and denote $x = u - \alpha v$ for a yet undetermined parameter $\alpha \in \mathbb{R}$. Note that the desired property $u = \alpha v$ is equivalent to $x = \vec{0}$. From (IP5) and bilinearity, we see that $x = \vec{0}$ is equivalent with $\langle x, x \rangle = 0$.

Let us calculate $\langle x, x \rangle$ for $x = u - \alpha v$ using bilinearity and the given inner product values among u and v, noting also $\langle v, u \rangle = \langle u, v \rangle$ by (IP1):

$$\begin{aligned} \langle x, x \rangle \ = \ \langle u - \alpha v,\ u - \alpha v \rangle \ &= \langle u, u \rangle - \alpha \langle u, v \rangle - \alpha \langle v, u \rangle + \alpha^2 \langle v, v \rangle \\ &= \frac{5}{9} + \frac{20}{21}\alpha + \frac{20}{49}\alpha^2 \\ &= 5\left(\frac{1}{3} + \frac{2}{7}\alpha\right)^2 \end{aligned}$$

We conclude that $x = \vec{0}$ if and only if the above expression vanishes, which occurs if and only if $\frac{1}{3} + \frac{2}{7}\alpha = 0$. The unique value of α for which this occurs is $\alpha = -\frac{7}{6}$.

In other words, we have

$$u \ = \ -\frac{7}{6} v$$

and no other value works, i.e., $u \neq \alpha v$ for $\alpha \neq -\frac{7}{6}$.

(1) stated bilinearity as a consequence of the defining properties (or correct use)
(2) correctly argued that $u = \alpha v$ is equivalent with $\langle u - \alpha v,\ u - \alpha v \rangle = 0$
(3) used bilinearity to calculate $\langle u - \alpha v,\ u - \alpha v \rangle$
(4) justifications: symmetry $\langle v, u \rangle = \langle u, v \rangle$ and no zero divisors
(5) solved the equation for α
(6) concluded correctly about the original question

Problem 4. **(6 pts)**

Let (X, d) be a metric space. Suppose that $K_1, K_2, K_3, \ldots \subset X$ are compact subsets of X. Consider the set

$$A = \left\{ x \in X \;\middle|\; x \in K_n \text{ for all } n \in \mathbb{N} \right\}.$$

Show that A is compact.

Solution. The notation simplifies by noting that A is defined so that it is precisely the intersection of the family $(K_n)_{n\in\mathbb{N}}$ of compact sets, $A = \bigcap_{n\in\mathbb{N}} K_n$.

To prove that A is (sequentially) compact, we must by definition show that any sequence in A has a convergent subsequence (converging in $A \subset X$). So let $(x_j)_{j\in\mathbb{N}}$ be an arbitrary sequence in A. For any $j \in \mathbb{N}$, since $x_j \in A = \bigcap_{n\in\mathbb{N}} K_n$, we have $x_j \in K_n$ for all $n \in \mathbb{N}$. In particular the sequence $(x_j)_{j\in\mathbb{N}}$ is also a sequence in K_n, for any $n \in \mathbb{N}$. Let us first consider just $n = 1$, and note that by the assumed compactness of K_1, the sequence $(x_j)_{j\in\mathbb{N}}$ in K_1 has some convergent subsequence $(x_{\varphi(j)})_{j\in\mathbb{N}}$, with limit $x' = \lim_{j\to\infty} x_{\varphi(j)} \in K_1 \subset X$. We will show that this subsequence $(x_{\varphi(j)})_{j\in\mathbb{N}}$ is convergent also in $A = \bigcap_{n\in\mathbb{N}} K_n$. For this, it suffices to show that its limit belongs to A, i.e., that $x' \in A$ (since the distances used in $A \subset X$ and $K_1 \subset X$ are the same, both induced by the metric on X).

Now for any $n \in \mathbb{N}$, since $(x_{\varphi(j)})_{j\in\mathbb{N}}$ is a sequence in K_n (by an observation above), by compactness of K_n, we know that some subsequence of $(x_{\varphi(j)})_{j\in\mathbb{N}}$ converges to some $x^{(n)} \in K_n$. But then in X, that subsequence has as its limits both $x^{(n)} \in K_n \subset X$ and $x' \in K_1 \subset X$ (as a subsequence of a sequence converging to x'), so by uniqueness of limits we have $x' = x^{(n)}$. In particular we get $x' \in K_n$. Since this holds for all n, we conclude that $x' \in \bigcap_{n\in\mathbb{N}} K_n = A$. This shows that the limit x' of the subsequence $(x_{\varphi(j)})_{j\in\mathbb{N}}$ is in A.[2]

This establishes that the sequence $(x_j)_{j\in\mathbb{N}}$ in A has a convergent subsequence (converging in $A \subset X$). Since the sequence $(x_j)_{j\in\mathbb{N}}$ in A was arbitrary, this proves (sequential) compactness of A.

(1) definition of compactness (sequential compactness or covering compactness)
(2) start by picking an arbitrary sequence in A (or an arbitrary open cover)
(3) noted that the sequence is also a sequence in K_n for any n
(4) used assumed compactness of (e.g.) K_1 to extract a convergent subsequence
(5) correctly argued that the limit of the subsequence is in every K_n
(6) concluded that the chosen subsequence converges also in A

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[2] Another correct argument leading to the conclusion of this paragraph is the following. Note that the compact sets $K_n \subset X$ are closed (by a theorem in the course), and as the intersection $A = \bigcap_{n\in\mathbb{N}} K_n$ of closed sets, A is itself closed (by a theorem in the course). Then by a characterization of closedness (a theorem in the course), the sequence $(x_{\varphi(j)})_{j\in\mathbb{N}}$ in A which converges in X must in fact have its limit in $A \subset X$.

Problem 5. **(6 pts)**
Let

$$S = \Big\{(x,y) \in \mathbb{R}^2 \;\Big|\; x > 0 \text{ and } y < 0\Big\} \subset \mathbb{R}^2,$$

and let $f\colon S \to \mathbb{R}$ and $g\colon S \to \mathbb{R}$ be two continuous functions. Assume that there exists points $z, w \in S$ such that $f(z) < g(z)$ and $f(w) > g(w)$. Show that there exists a point $u \in S$ such that $f(u) = g(u)$.

Solution. The key is to observe that S is connected, and in fact path-connected (path-connectedness makes the proof somewhat more concrete, although a correct proof based on connectedness can also be given).

Path connectedness means that for any two points $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in S$, there exists a continuous path $\gamma\colon [0,1] \to S$ such that $\gamma(0) = z_1$ and $\gamma(1) = z_2$. To prove path connectedness of S, let $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in S$ be two arbitrary points in S, and construct γ by the formula

$$\gamma(t) \;=\; \big(x_1 + t(x_2 - x_1),\; y_1 + t(y_2 - y_1)\big) \qquad \text{for } t \in [0,1].$$

It is clear from the definition that $\gamma(t) \in \mathbb{R}^2$ (we have given two real coordinates for it). For $t \in [0,1]$, we check that in fact $\gamma(t) \in S \subset \mathbb{R}^2$ as follows. Since $z_1, z_2 \in S$, we have $x_1, x_2 > 0$ and $y_1, y_2 < 0$. Then for $t \in [0,1]$, we write[3]

$$x_1 + t(x_2 - x_1) \;=\; \underbrace{(1-t)}_{\geq 0}\,\underbrace{x_1}_{>0} + \underbrace{t}_{\geq 0}\,\underbrace{x_2}_{>0} \;\overset{(*)}{>}\; 0$$

$$y_1 + t(y_2 - y_1) \;=\; \underbrace{(1-t)}_{\geq 0}\,\underbrace{y_1}_{<0} + \underbrace{t}_{\geq 0}\,\underbrace{y_2}_{<0} \;\overset{(*)}{<}\; 0,$$

which are the defining properties of points of S, so $\gamma(t) \in S$. Therefore indeed γ is a function $[0,1] \to S$. Clearly $\gamma(0) = z_1$ and $\gamma(1) = z_2$. Also, γ is $\|z_2 - z_1\|$-Lipschitz (seen in the course), and therefore continuous.[4]

Using path-connectedness of S, we will address the problem itself. As in the problem statement, assume that $f, g\colon S \to \mathbb{R}$ are continuous and that $z, w \in S$ are such that $f(z) < g(z)$ and $f(w) > g(w)$. First pick a path from z to w in S, i.e., a continuous function $\gamma\colon [0,1] \to S$ such that $\gamma(0) = z$ and $\gamma(1) = w$. Construct also a new function

$$h\colon\; S \to \mathbb{R} \qquad\qquad h(u) = g(u) - f(u) \quad \text{for } u \in S.$$

In the course we have seen that (finite) pointwise sums and scalar multiples of real-valued continuous functions remain continuous, so also $h = g - f$ is continuous $S \to \mathbb{R}$. Now the composed function

$$\begin{tikzcd} {[0,1]} \arrow[r, "\gamma"'] \arrow[rr, bend left, "h\circ\gamma"] & S \arrow[r, "h"'] & \mathbb{R} \\ t \arrow[r, mapsto, "\gamma"'] & \gamma(t) \arrow[r, mapsto, "h"'] & h\big(\gamma(t)\big), \end{tikzcd}$$

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[3] (*): at least one of the coefficients t and $1-t$ is non-zero, so we obtain strict inequalities here.

[4] Alternatively, the continuity of γ is seen from the fact that the component functions are polynomial functions and as such continuous.

is a continuous function $h \circ \gamma \colon [0,1] \to \mathbb{R}$, as the composition of the continuous functions γ and h. We have

$$\begin{aligned} (h \circ \gamma)(0) \;&=\; h\big(\gamma(0)\big) \;=\; h(z) \;=\; g(z) - f(z) \;>\; 0 \\ (h \circ \gamma)(1) \;&=\; h\big(\gamma(1)\big) \;=\; h(w) \;=\; g(w) - f(w) \;<\; 0. \end{aligned}$$

By the intermediate value theorem (Bolzano's theorem) we have that since the values of the continuous function $h \circ \gamma \colon [0,1] \to \mathbb{R}$ at the two endpoints of the interval $[0,1]$ have opposite signs, there exists an $s \in (0,1)$ such that $(h \circ \gamma)(s) = 0$. Let $u = \gamma(s) \in S$. Then unraveling the definitions, we have

$$0 \;=\; (h \circ \gamma)(s) \;=\; h\big(\gamma(s)\big) \;=\; h(u) \;=\; g(u) - f(u),$$

from which we solve

$$f(u) = g(u).$$

We have thus shown the existence of a point $u \in S$ with the desired property.

1. Observed and justified (at least by a picture) the path-connectedness of S.
2. For z, w as in the problem, picked a path connecting them.
3. Constructed $h = g - f$ and $h \circ \gamma \colon [0,1] \to \mathbb{R}$.
4. Observed and justified the continuity of $g - f$ and of the composition $h \circ \gamma$.
5. Applied Bolzano's theorem to find an $s \in [0,1]$ such that $(h \circ \gamma)(s) = 0$.
6. Concluded that $u = \gamma(s)$ has the desired property $f(u) = g(u)$.

Problem 6. **(6 pts)**

(a) For $n \in \mathbb{N}$, let $f_n \colon [0, \infty) \to \mathbb{R}$ be the function given by

$$f_n(x) = \frac{n}{1 + nx + n^2\,(x^2 - x)^2} \qquad \text{for } x \in [0, \infty).$$

Does the function sequence $(f_n)_{n \in \mathbb{N}}$ converge pointwise? Does the function sequence $(f_n)_{n \in \mathbb{N}}$ converge uniformly?

(b) For $n \in \mathbb{N}$, let $g_n \colon [1, \infty) \to \mathbb{R}$ be the function given by

$$g_n(x) = \frac{n}{1 + nx + n^2\,(x^2 - x)^2} \qquad \text{for } x \in [1, \infty).$$

Does the function sequence $(g_n)_{n \in \mathbb{N}}$ converge pointwise? Does the function sequence $(g_n)_{n \in \mathbb{N}}$ converge uniformly?

(c) For $n \in \mathbb{N}$, let $h_n \colon [2, \infty) \to \mathbb{R}$ be the function given by

$$h_n(x) = \frac{n}{1 + nx + n^2\,(x^2 - x)^2} \qquad \text{for } x \in [2, \infty).$$

Does the function sequence $(h_n)_{n \in \mathbb{N}}$ converge pointwise? Does the function sequence $(h_n)_{n \in \mathbb{N}}$ converge uniformly?

Solution. Recall the definitions of pointwise and uniform convergence (of real-valued functions, for concreteness; in this problem the codomain of all functions is $\mathbb{R}$). Let $(\phi_n)_{n \in \mathbb{N}}$ be a sequence real-valued functions $\phi_n \colon X \to \mathbb{R}$ on a common domain X. The sequence $(\phi_n)_{n \in \mathbb{N}}$ converges pointwise to a limit function $\phi \colon X \to \mathbb{R}$ if

$$\forall x \in X : \qquad \lim_{n \to \infty} \phi_n(x) = \phi(x). \tag{1}$$

The sequence $(\phi_n)_{n \in \mathbb{N}}$ converges uniformly to a limit function $\phi \colon X \to \mathbb{R}$ if

$$\lim_{n \to \infty} \sup_{x \in X} \big|\phi_n(x) - \phi(x)\big| \;=\; 0. \tag{2}$$

Also recall the following results from the course:

- Uniform convergence implies pointwise convergence to the same limit function.
- The limit ϕ of a uniformly convergent sequence of continuous functions is continuous.

(a): We claim that the given sequence $(f_n)_{n \in \mathbb{N}}$ of functions $f_n \colon [0, \infty) \to \mathbb{R}$ does not converge pointwise. It will follow that it can not converge uniformly either (since uniform convergence implies pointwise convergence). Consider the point $x = 0$. The value of the n:th function at $x = 0$ is

$$f_n(0) = \frac{n}{1 + n\,0 + n^2(0^2 - 0)^2} = \frac{n}{1} = n.$$

Now $\lim_{n \to \infty} f_n(0) = \lim_{n \to \infty} n$ does not exist (in $\mathbb{R}$), so we do not have convergence (1) of the values at $x = 0$, and therefore no pointwise convergence of $(f_n)_{n \in \mathbb{N}}$.

(b): We claim that the given sequence $(g_n)_{n \in \mathbb{N}}$ of functions $g_n \colon [1, \infty) \to \mathbb{R}$ converges pointwise but not uniformly. To verify pointwise convergence, let $x \in [1, \infty)$.

Consider separately the cases $x = 1$ and $x > 1$. If $x = 1$, we have

$$g_n(1) = \frac{n}{1 + n\,1 + n^2(1^2 - 1)^2} = \frac{n}{1+n} = \frac{1}{1+\frac{1}{n}} \longrightarrow \frac{1}{1} = 1 \qquad \text{as } n \to \infty.$$

If $x > 1$, we have $x^2 - x = x(x-1) > 0$ and therefore $(x^2 - x)^2 > 0$. We then get

$$g_n(x) = \frac{n}{1 + n\,x + n^2(x^2 - x)^2} = \frac{\frac{1}{n}}{(x^2-x)^2 + \frac{x}{n} + \frac{1}{n^2}} \longrightarrow \frac{0}{(x^2-x)^2} = 0 \qquad \text{as } n \to \infty.$$

We have shown convergence of values at all points, and so get pointwise convergence to the limit function

$$g(x) = \begin{cases} 1 & \text{for } x = 1 \\ 0 & \text{for } x > 1. \end{cases}$$

This limit function g is not continuous, so we cannot have uniform convergence of $(g_n)_{n\in\mathbb{N}}$: each of the functions g_n is continuous as a rational function, and the limit function of a uniformly convergent sequence would then be continuous.

(c): We claim that the given sequence $(h_n)_{n\in\mathbb{N}}$ of functions $h_n \colon [2,\infty) \to \mathbb{R}$ converges uniformly to the limit function h, which is the zero function: $h(x) = 0$ for all $x \in [2,\infty)$. From uniform convergence, also pointwise convergence will follow. To verify the uniform convergence, observe that for any $x \in [2,\infty)$, we have $x \geq 2$ and $x^2 - x = x(x-1) \geq 2$ and therefore $(x^2-x)^2 \geq 4$. We then get

$$\big|h_n(x) - h(x)\big| = \big|h_n(x) - 0\big| = h_n(x) = \frac{n}{1 + n\,x + n^2(x^2-x)^2} \leq \frac{n}{1 + 2n + 4n^2}.$$

Since this holds for any $x \in [2,\infty)$, we get

$$0 \;\leq\; \sup_{x\in[2,\infty)} \big|h_n(x) - h(x)\big| \;\leq\; \frac{n}{1+2n+4n^2}. = \frac{\frac{1}{n}}{4 + \frac{2}{n} + \frac{1}{n^2}} \longrightarrow \frac{0}{4} = 0 \qquad \text{as } n \to \infty.$$

This shows (by an application of the squeeze theorem) the uniform convergence (2)

$$\lim_{n\to\infty} \sup_{x\in[2,\infty)} \big|h_n(x) - h(x)\big| \;=\; 0.$$

(1) Stated and verified that $(f_n)_{n\in\mathbb{N}}$ does not converge pointwise.
(2) Justified that $(f_n)_{n\in\mathbb{N}}$ cannot converge uniformly either.
(3) Stated and verified that $(g_n)_{n\in\mathbb{N}}$ converges pointwise.
(4) Justified that $(g_n)_{n\in\mathbb{N}}$ cannot converge uniformly.
(5) Observed pointwise convergence of $(h_n)_{n\in\mathbb{N}}$ (consequence of uniform conv.).
(6) Stated and verified that $(h_n)_{n\in\mathbb{N}}$ converges uniformly.