Exam

15.04.2024

- You need to justify all of your answers.
- You can use a non-graphing calculator.
- You can use one two-sided cheat sheet (maximal DIN A4).

Good luck!

Problem 1 (15 points)

Consider a linear program (P) and corresponding optimal tableau (T).

$$
(P) \quad \begin{aligned} \max\ & 2x_1 + 3x_2 - 2x_3 && (1)\\ \text{s.t.}\ & -x_1 + 2x_2 + x_3 = 1 && (2)\\ & -4x_1 + 2x_2 + x_4 = 4 \\ & 5x_1 - x_2 + x_5 = 3 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}
$$

(T)

	x_1	x_2	x_3	x_4	x_5	Sol
$-\bar{c}$	0	0	$\frac{35}{9}$	0	$\frac{7}{9}$	$\frac{38}{9}$
x_2	0	1	$\frac{5}{9}$	0	$\frac{1}{9}$	$\frac{8}{9}$
x_4	0	0	$-\frac{2}{3}$	1	$\frac{2}{3}$	$\frac{16}{3}$
x_1	1	0	$\frac{1}{9}$	0	$\frac{2}{9}$	$\frac{7}{9}$

a) (4 points) Bring (P) into tableau form and identify a basis. Is this a feasible tableau?

b) (3 points) Argue why tableau (T) is optimal, identify an optimal basis, an optimal solution and the optimal objective function value.
Hint: *You do not need to show how tableau (T) is derived.*

c) (4 points) Suppose that in (P) the objective function (1) is changed to

$$2x_1 + 3x_2 - 2x_3 + \Delta_4 x_4.$$

Compute the minimum and maximum value of Δ_4 such that the basis remains optimal.

d) (4 points) Suppose that in (P) constraint (2) is changed to

$$-x_1 + 2x_2 + x_3 = 1 + \Delta_1.$$

Compute the minimum and maximum value of Δ_1 such that the basis remains optimal.
Hint: *You can use the tableau from a).*

Problem 2 (20 points)

You want to design $40m^2$ of garden with grass, vegetable patches and flower beds such that at least $10m^2$ are covered by grass and there is twice as much area for vegetable patches as for flower beds. Currently, you have seeds for $15m^2$ grass, $10m^2$ of vegetable patches and $7m^2$ of flower beds. You can buy further seeds, where grass costs $5€/m^2$, vegetables cost $3€/m^2$ and flowers $7€/m^2$. How can you design your garden as cheaply as possible?
Hint: *You do not need to solve this problem.*

a) (10 points) Model the problem as a linear program.

b) (5 points) Additionally, there are enough wood chips available for up to $2m^2$ of pathways. Renting a machine to prepare the ground costs 10€. Derive a model that includes the option to add pathways as part of the garden area based on the model from a).
Hint: *The new model does not have to be a linear program.*

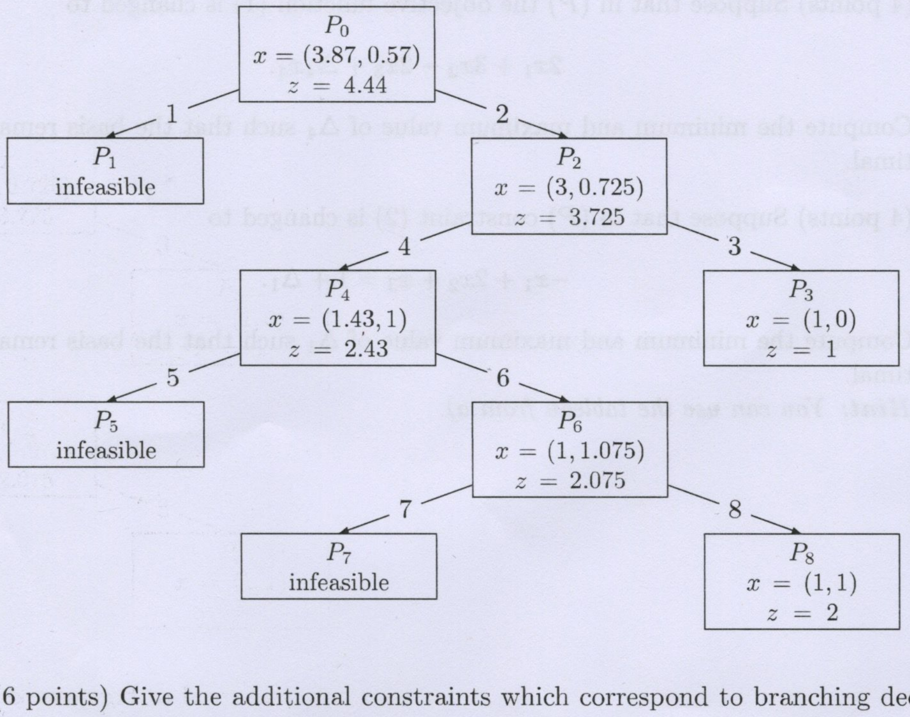
c) (5 points) Suppose you want to plant the vegetables in a rectangle and add a fence for $1€/m$. How do you have to adapt model a) to accommodate for this change?
Hint: *The new model does not have to be a linear program.*

Problem 3 (10 points)

Consider the optimization problem

$$\max x_1 + x_2$$
$$\text{s.t. } 0.175x_1 + x_2 \leq 1.25$$
$$0.2x_1 - x_2 \leq 0.2$$
$$x_1, x_2 \geq 0$$

with the following branch-and-bound tree. The subproblems are solved in the order $P_0, P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8$.



a) (6 points) Give the additional constraints which correspond to branching decisions on the edges 1 to 8. What is the optimal solution of the problem?

b) (4 points) Identify the best known lower bound after solving each subproblem P_i, $i \in \{0, \ldots, 8\}$.

Problem 4 (30 points)

Consider the unconstrained optimization problem

$$\min \quad f(x_1, x_2) = \frac{3}{2}x_1^2 - x_1 x_2 + (3 - x_2)^2.$$

0 0 9

a) (10 points) Compute an optimal solution x^* analytically and give the corresponding necessary and sufficient optimality conditions.

b) (10 points) Compute one step of the gradient descent method with $x_0 = (0,0)^\top$, $\varepsilon = 0.001$ and optimal stepwidth λ. Justify why the resulting solution is or is not optimal without relying on the solution x^* from a).

c) (10 points) Compute one step of Newton's method with $x_0 = (0,0)^\top$, $\varepsilon = 0.001$ and optimal stepwidth λ. Justify why the resulting solution is or is not optimal without relying on the solution x^* from a).

Hint: *The eigenvalues of* $A = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$ *are approximately 3.62 and 1.38,* $A^{-1} = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{5} \end{pmatrix}$.

Hint: *If you cannot compute the optimal stepwidth, you can use* $\lambda = 1$ *in your computations. Note that this will not result in full points.*

Problem 5 (15 points)

Consider the following problem

$$\begin{aligned} \min\ & 2x_1^2 - 3x_1 - 4x_2 + 5x_2^2 \\ \text{s.t. } & x_2^2 - 9 \le 0 \\ & 4x_1 - x_2 - 2 = 0. \end{aligned}$$

a) (3 points) Argue whether Slater's constraint qualification conditions hold.

b) (4 points) Formulate the KKT conditions.

c) (8 points) Find a point that satisfies the KKT conditions.

Problem 6 (10 points)

Consider the optimization problems (P_1) and (P_2).

$$\begin{aligned} (P_1)\ \max\ & f_1(x_1, x_2) = c_1 x_1 + c_2 x_2 \\ \text{s.t. } & a_1 x_1 + a_2 x_2 \le b_1 \\ & d_1 x_1 + d_2 x_2 \le b_2 \\ & x_1, x_2 \in \{0, 1\} \end{aligned}$$

$$\begin{aligned} (P_2)\ \max\ & f_2(x_1, x_2) = c_1 x_1^2 + c_2 x_2^2 + (b_2 - d_1 x_1 - d_2 x_2) \\ \text{s.t. } & a_1 x_1 + a_2 x_2 \le b_1 \\ & x_1 \le 1 \\ & x_2 \le 1 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Let $\mathcal{F}_1$ be the feasible set of (P_1) and $\mathcal{F}_2$ be the feasible set of (P_2). Show that (P_2) is a relaxation of (P_1), i.e.,

a) (5 points) $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and

b) (5 points) $f_2(x_1, x_2) \ge f_1(x_1, x_2)$ for all $(x_1, x_2) \in \mathcal{F}_1$.