Mathematics
Aalto University

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MS-E1653 Finite Element Method

Exam 16.4.2024

Please fill in clearly *on every sheet* the data on you and the examination. On *Examination code* mark the course code, title and text mid-term or final examination.

You have two options

1. Solve all problems. Grade is based only on the exam.
2. Solve any three problems. Grade is based on exercise points + exam points. To choose this option, you must have completed the final project.

The exam time is three hours (3h). No electronic calculators or materials are allowed.

1. Consider the partition $\{0, \frac{1}{2}, 1\}$ and the associated first order FE-space $\widehat{V}_h$. Denote the second element by $I_2 := (\frac{1}{2}, 1)$.
   (a) Give formula for the hat basis function $\hat{\varphi}_2$. (1p)
   (b) Let $\widehat{A}$ be the system matrix corresponding to $\widehat{V}_h$ and weak form $\int_0^1 u'v'$. Compute entry $\widehat{A}_{2,2}$. (2p)
   (c) Let $\widehat{b}$ be the load vector corresponding to $\widehat{V}_h$ and linear functional $\int_0^1 v$. Compute entry $\widehat{b}_2$. (2p)
   (d) Use $\widehat{V}_h$ to approximately solve the problem: find $u \in H_0^1(0,1)$ satisfying
   $$\int_0^1 u'v' = \int_0^1 v \quad \text{for all } v \in H_0^1(0,1). \qquad (1p)$$

2. Let V be a Hilbert space, $a : V \times V \to \mathbb{R}$ be a symmetric, elliptic and continuous bilinear form, $\|\cdot\|_E^2 = a(\cdot,\cdot)$, $L : V \to \mathbb{R}$ a continuous linear functional, and $J(v) = \frac{1}{2}a(v,v) - L(v)$. Let $u \in V$ satisfy $a(u,v) = L(v)$ for any $v \in V$. Show that
   (a) $J(u+v) > J(u)$ for any $v \in V$, $v \neq 0$. (2p)
   (b) $\|u - v\|_E^2 = 2(J(v) - J(u))$ for any $v \in V$. (2p)
   (c) Let spaces $\{V_1, V_2, V_3\} \subset V$ and $u_i \in V_i$ satisfy
   $$a(u_i, v_i) = L(v_i) \quad \text{for each } v \in V_i \quad \text{and } i \in \{1,2,3\}.$$
   Assume that $J(u_1) = 1$, $J(u_2) = 0.3$, $J(u_3) = -0.1$, and $J(u) = -0.5$. Which of the solutions $\{u_i\}_{i=1}^3$ is the most accurate approximation of u? Justify your answer. (2p)

3. Let the finite element mesh $\mathcal{T}$ be such that

$$p = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array} \quad \text{and} \quad t = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 3 & 5 \\ 4 & 5 & 5 & 6 \end{bmatrix} \tag{1}$$

(a) Draw the mesh $\mathcal{T}$. (1p)

(b) Compute affine mapping from the reference element to element 4. (2p)

(c) Use the affine mapping to define $\hat{\varphi}_3|_{T_4}$ and $\nabla\hat{\varphi}_3|_{T_4}$. (3p)

4. Let V_h be the first order FE-space related to partition $\{x_j\}_{j=1}^N$ of interval I. Assume there exists ρ independent of N and h such that

$$\min_{i\in\{1,\ldots,N-1\}} (x_{i+1} - x_i) \geq \rho h.$$

Prove *the inverse inequality*: there exists a constant C dependent on ρ but independent of v_h and h such that

$$\|v_h'\|_{L^2(I)} \leq Ch^{-1}\|v_h\|_{L^2(I)} \quad \text{for all } v_h \in V_h. \tag{2}$$

Use the scaling argument and the following result:

$$\|\hat{p}'\|_{L^2(0,1)} \leq \widehat{C}\|\hat{p}\|_{L^2(0,1)} \quad \text{for all } \hat{p} \in P^1(0,1),$$

where constant $\widehat{C}$ is independent of $\hat{p}$.

Scaling Argument Let $a < b$, $k \in \{0,1,\ldots\}$, $h_I := (b-a)$, and $r : (0,1) \mapsto (a,b)$ be defined as $r(\hat{t}) := (b-a)\hat{t} + a$. In addition, let $v \in H^k(a,b)$ and $\hat{v} \in H^k(0,1)$ be defined as $\hat{v}(\hat{t}) := v(r(\hat{t}))$. Then there holds that

$$\left\|\frac{d^{(k)}v}{dt^{(k)}}\right\|_{L^2(a,b)} = h_I^{(1-2k)/2}\left\|\frac{d^{(k)}\hat{v}}{d\hat{t}^{(k)}}\right\|_{L^2(0,1)}. \tag{3}$$

Here $\frac{d^{(0)}v}{dt^{(0)}} = v$ and $\frac{d^{(0)}\hat{v}}{d\hat{t}^{(0)}} = \hat{v}$.