Aalto University - School of Science, Department of Mathematics and System Analysis

MS-A0311 - Differential and integral calculus 3 Rogovin/Brummet
Course exam and general exam 17.4.2024 klo 9.00–12.00.

Course exam: the five best problems will be included in the evaluation.
General exam: Do all six problems.
All participants of the lecture course in the period IV/2024 can do all six problems. The evaluation is calculated with that option that gives the highest score: either "five best problems + exercise points" or "6 problems only".

No calculators or notes of any kind are allowed.

Each problem is worth 6 points.

1. Consider the vector fields $\mathbf{F}(x, y, z) = (2x+3z)\mathbf{i}+(4x-2y)\mathbf{j}+(5x-6y)\mathbf{k}$ and $\mathbf{G}(x, y, z) = xyz\,\mathbf{i}$. Check for both vector fields (the notation written only for $\mathbf{F}$, but do the same also for $\mathbf{G}$):
   (a) Is the vector field solenoidal i.e. $\nabla \cdot \mathbf{F} = 0$ everywhere (so the field has no sources or sinks)?
   (b) Is the vector field irrotational i.e. $\nabla \times \mathbf{F} = \mathbf{0}$ everywhere?
   (c) Is the vector field complex lamellar i.e. $\mathbf{F} \cdot (\nabla \times \mathbf{F}) = 0$ everywhere?

2. The temperature T of the ball
$$B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq R^2\}$$
decreases linearly with respect to the spherical coordinate variable r from the value 6000 of the center to the value 0 of the surface, i.e. $T(r) = 6000(1 - r/R)$, where $0 \leq r \leq R$. Calculate the mean temperature of the ball
$$\bar{T} = \frac{1}{V}\iiint_B T\,dV.$$
The volume of the ball is $V = 4\pi \cdot R^3/3$.

3. Let C be a curve with a parametrization $\mathbf{r}(t) = (3t, 4t^2, 3t^2)$, where $-\pi \leq t \leq \pi$, and let $\mathbf{F}$ be the vector field
$$\mathbf{F}(x, y, z) = (2xyz + \sin x,\ x^2 z,\ x^2 y).$$
   (a) Find a potential function φ for $\mathbf{F}$.
   (b) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

4. Calculate $\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{N}}\,dS$, where $\mathbf{F}(x, y, z) = (xz + yz^2 + x,\ xyz^3 + y,\ x^2z^4)$, and $\hat{\mathbf{N}}$ is the unit normal field for S pointing away from the origin, and
$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \text{ and } z \geq 0\}.$$

5. Calculate the flux of the vector field $\mathbf{F}(x, y, z) = (x,\ 2y,\ z^2)$ out of the cylinder

$$D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 9,\ 0 \leq z \leq 1\}$$

through its surface.

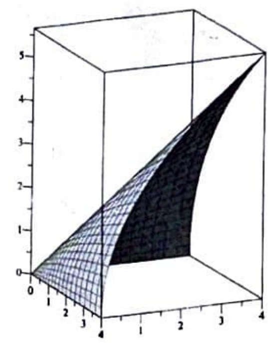
6. The architect designed a canopy for an open-air concert stage (see the picture below). The shape of the surface is described by a parametrization

$$\mathbf{r}(u, v) = (u^2,\ v^2,\ \sqrt{2}\, uv), \quad 0 \leq u \leq 2,\ 0 \leq v \leq 2.$$

(a) Calculate the normal vector for the surface at $\mathbf{r}(u, v)$. (2p)

(b) Show that the angle of inclination[1] for the canopy on the line that correspondence to the parameter values $u = v \in [0, 2]$ is 45°. (1p)

(c) Show that the area element of the surface $dS = 2\sqrt{2}\,(u^2 + v^2)\, du dv$ and calculate the surface are of the canopy. (3p)



Formulas without explanation:

- $x^2/a^2 + y^2/b^2 = 1 \longleftrightarrow x = a\cos t,\ y = b\sin t$
- $\nabla \cdot (\nabla f) = \Delta f,\ \nabla \times \nabla f = \bar{0},\ \nabla \cdot (\nabla \times \mathbf{F}) = 0$
- $\nabla \cdot (f\mathbf{F}) = \nabla f \cdot \mathbf{F} + f(\nabla \cdot \mathbf{F}),\ \nabla \times (f\mathbf{F}) = \nabla f \times \mathbf{F} + f(\nabla \times \mathbf{F})$
- $\oint_{\partial D} F_1\, dx + F_2\, dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$
- In the following formulas $\hat{\mathbf{N}}$ = unit normal.
- $\iiint_D \nabla \cdot \mathbf{F}\, dV = \oiint_{\partial D} \mathbf{F} \cdot \hat{\mathbf{N}}\, dS$
- $\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{N}}\, dS = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \oint_{\partial S} F_1\, dx + F_2\, dy + F_3\, dz$
- (r, φ, θ): $x = r\sin(\varphi)\cos(\theta),\ y = r\sin(\varphi)\sin(\theta),\ z = r\cos(\varphi),\ dV = r^2\sin(\varphi)\, dr\, d\varphi\, d\theta$
- (ρ, θ, z): $x = \rho\cos(\theta),\ y = \rho\sin(\theta),\ z = z,\ dV = \rho\, d\rho\, d\theta\, dz$
- $\sin(\pi/6) = \cos(\pi/3) = 1/2,\ \sin(\pi/3) = \cos(\pi/6) = \sqrt{3}/2,\ \sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2},$ $\sin 0 = \cos(\pi/2) = 0,\ \sin(\pi/2) = \cos 0 = 1,\ \sin\pi = 0,\ \cos\pi = -1,\ \sin^2 x + \cos^2 x = 1,$ $\sin(2x) = 2\sin x\cos x,\ \cos(2x) = 2\cos^2 x - 1 = 1 - 2\sin^2 x.$

Note 1: Answering the course feedback gives one course point!
Note 2: You can retake the course exam when there is a general exam at June. On that retake the exercise points of the course are valid. To retake the exam you have to enroll to the exam through Sisu.

[1] the angle at which a surface is tilted or inclined from the horizontal