



PHYS-E0420 Many-Body Quantum Mechanics
Midterm Exam 21.02.2024

1. Write a mini-essay (0.5-1 page each) about the following two topics:
 - a) Schrödinger, Heisenberg, and interaction pictures. What is meant by each of them?
 - b) Bosons and fermions: fundamental and physical differences, and their treatment in the second quantization technique
2. Derive the Fermi golden rule for the (absorption) transition rate from the initial state $|\psi(t=0)\rangle = |i\rangle$ to a continuous part of the spectrum.

- a) Start from the first-order time-dependent perturbation theory result

$$\langle f|\psi(t)\rangle \approx \frac{1}{\hbar} \int_0^t dt' \langle f|H'(t')|i\rangle e^{i\omega_{fi}t'}$$

where $|i\rangle$ and $|f\rangle$ are the initial and final states of the system, $\omega_{fi} = (E_f - E_i) / \hbar$, and $H'(t) = Ae^{-i\omega t} + A^\dagger e^{i\omega t}$ is a harmonic perturbation. Derive the result for resonant absorption probability

$$P_{if}(t) = \frac{1}{\hbar^2} |\langle f|A|i\rangle|^2 \left[\frac{\sin \left[\frac{1}{2} (\omega - \omega_{fi}) t \right]}{\frac{1}{2} (\omega - \omega_{fi})} \right]^2, \quad \text{for } \omega \approx \omega_{fi}$$

You can assume that the perturbation is long lasting, such that $|\omega t| \gg 1$.

- b) Using the above transition probability, write the total transition rate out from the initial state $|i\rangle$ when there are several possible final states $|f_n\rangle$ with energies $E_{f,n} = \hbar\omega_n$. You can assume that the coupling $\langle f|A|i\rangle$ is equal for all final states $|f\rangle = |f_n\rangle$. Now write the total transition rate when the final states form a continuum with constant density of states $g(E) = g$.
- c) Solve the total transition rate and derive the Fermi golden rule. Since off-resonant (non-energy-conserving) processes are strongly suppressed, you are able to do some approximations. The following relation may also prove useful

$$\int_{-\infty}^{\infty} \left[\frac{\sin x}{x} \right]^2 dx = \pi.$$

3. The Hamiltonian in the field operator formulation is

$$H = \int d^3x \left(\frac{\hbar^2}{2m} \nabla \psi^\dagger(\mathbf{x}) \nabla \psi(\mathbf{x}) + U(\mathbf{x}) \psi^\dagger(\mathbf{x}) \psi(\mathbf{x}) \right) + \frac{1}{2} \int d^3x \int d^3x' \psi^\dagger(\mathbf{x}) \psi^\dagger(\mathbf{x}') V(\mathbf{x}, \mathbf{x}') \psi(\mathbf{x}') \psi(\mathbf{x}),$$

where U is an external potential, m the mass of the particle, and V the interaction potential. Starting from the Heisenberg equation of motion

$$\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}, t) = -[H, \psi(\mathbf{x}, t)] = -e^{iHt/\hbar} [H, \psi(\mathbf{x}, 0)] e^{-iHt/\hbar}$$

derive the equation of motion for the field operator:

$$\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}, t) = \left(-\frac{\hbar^2 \nabla^2}{2m} + U(\mathbf{x}) \right) \psi(\mathbf{x}, t) + \int d^3x' \psi^\dagger(\mathbf{x}', t) V(\mathbf{x}, \mathbf{x}') \psi(\mathbf{x}', t) \psi(\mathbf{x}, t).$$

How does the form of this equation relate to the name "second quantization"?

4. Coherent state $|\alpha\rangle$ can be created by acting on the vacuum $|0\rangle$ with a displacement operator $D(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a})$ (α is a complex number). Using this, compute the coherent state expectation values for the annihilation operator \hat{a} and for the energy $\hat{H} = \hbar\omega(\hat{a}^\dagger \hat{a} + 1/2)$.

Tip: Baker-Hausdorff relation

$$e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \dots$$