Final Exam I

[Don't panic looking at long questions; the description is sometimes long but that is meant to guide you toward the solution.]

Question 1 *(Concepts)*

Describe briefly:

(A) What are the main concepts introduced when discussing transmission lines? What effects have you seen related to transmission line that appear in stark contrast to lumped (discrete) circuit elements?

(B) What is Meissner effect? Can it be derived from the assumption of perfect conductivity?

Question 2 *(Lumped electrical circuits)*

Consider the circuit shown in Fig. 1, where an ac voltage $V(t) = V_0 \cos(\omega t)$ is present at the input. The numerical values of these components are $R = 5\ \text{k}\Omega$, $C = 10$ pF (pico $= 10^{-12}$), and the voltage amplitude is $V_0 = 100$ mV.

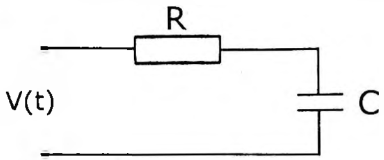

Figure 1 – A circuit consisting of a resistor and a capacitor with a voltage $V(t)$ at the input.

(a) What is the phasor of the current that flows in the circuit? Do you expect that the current would be larger at lower or at higher frequencies and why?

(b) Calculate the average dissipated power on the resitor as a function of frequency and plot it.

(c) What is the characteristic timescale of this circuit? What is the average power dissipated on the resistor for frequencies much larger than this timescale? Express this power in dBm. Comment on the results.

Question 3 *(Josephson effect)*

Consider a Josephson junction with critical current I_c, biased by a constant dc voltage V_{dc}.

(A) Write the current-phase Josephson relation. For $I_c = 1$ mA and voltage $V_{dc} = 10\ \mu$V plot the current and the Josephson inductance and calculate their oscillation period. How is this different from a regular inductor?

(B) We add now an ac component $V_{ac} \cos \omega t$, overlapping with the dc voltage. Obtain an expression of the current in harmonic components. Discuss what happens when $V_{dc} = m\Phi_0\omega/(2\pi)$, where m is an integer and Φ_0 is the magnetic flux quantum.

[Hint: You can use the expansion $e^{ia\sin x} = \sum_{n=-\infty}^{+\infty} J_n(a)e^{inx}$, where J_n is the Bessel function of the n^{th} kind, with the property $J_{-n}(a) = (-1)^n J_n(a)$. The phase-voltage Josephson relation is $d\varphi/dt = (2\pi/\Phi_0)V$, where $\Phi_0 = h/(2e)$, $h = 6.626 \times 10^{-34}$ Js, $e = 1.6 \times 10^{-19}$ C.]

Question 4 *(Lagrangian and Hamiltonian of a superconducting circuit)*

Consider the following Lagrangian:

$$L(\phi,\dot{\phi}) = \frac{1}{2}C_g(\dot{\phi}-V_g)^2 + \frac{1}{2}C_J\dot{\phi}^2 + E_J\cos\phi. \tag{1}$$

(a) What kind of quantum circuit does this Lagrangian represent? What are the generalized coordinates?

(b) Sketch a circuit diagram, labelling and defining all relevant parameters from the Lagrangian in the context of the physical quantum circuit.

(c) Set $V_g = 0$. Derive the Hamiltonian of the circuit using the Legendre transformation (see the hint). Express the Hamiltonian in terms of canonical momentum p_ϕ.

(d) Write down the equations of motion using Hamilton's equation.

Hint: $H = \sum_i \dot{x}_i p_i - L$, where x_i and $\dot{x}_i$ are the generalized coordinates. $p_i = \frac{\partial L}{\partial \dot{x}_i}$ is the conjugate momentum. Hamilton's equations: $\dot{x}_i = \frac{\partial H}{\partial p_i}$ and $\dot{p}_i = -\frac{\partial H}{\partial x_i}$.

Question 5 *(Light-matter interaction)*

Consider a quantum system where a single qubit (a perfect two-level system) is coupled to a microwave resonator. If the resonator and the qubit are designed to have the same frequency, then the quantum system can be described by a simple Jaynes-Cummings interaction Hamiltonian:

$$\hat{H}_{\text{int}} = \hbar g\left(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-\right), \tag{2}$$

where g describes the coupling strength, $\hat{a}$ and $\hat{a}^\dagger$ are the annhilation and creation operator respectively acting on harmonic modes of resonator, and $\hat{\sigma}_+$ and $\hat{\sigma}_-$ are the raising and lowering operators respectively for the qubit. We have the following identities:

$$\begin{aligned}
\hat{\sigma}_+|0\rangle &= |1\rangle \\
\hat{\sigma}_-|1\rangle &= |0\rangle \\
\hat{\sigma}_+|1\rangle &= \hat{\sigma}_-|0\rangle = 0 \\
\hat{a}|n\rangle &= \sqrt{n}|n-1\rangle \\
\hat{a}^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle
\end{aligned}$$

Here, the state of the cavity mode can be written in terms of the number-state basis $\{|n\rangle\}$, and the qubit state in terms of the computational basis $\{|0\rangle, |1\rangle\}$. The overall state of the system can therefore be described as a tensor product of these two subsystems, e.g., $|n,0\rangle = |n\rangle\otimes|0\rangle$.

(a) Show that the following relations hold:

$$\begin{aligned}
\hat{H}_{\text{int}}|n,0\rangle &= \hbar g\sqrt{n}|n-1,1\rangle \\
\hat{H}_{\text{int}}|n,1\rangle &= \hbar g\sqrt{n+1}|n+1,0\rangle
\end{aligned}$$

[Hint: Remember that in the combined system, the cavity operators only act on the cavity states and the qubit operators only act on the qubit states, e.g., $\hat{a}\hat{\sigma}_+|n,0\rangle = \hat{a}|n\rangle\otimes\hat{\sigma}_+|0\rangle$.]

(b) Consider now the case where the cavity starts in the vacuum state $|n=0\rangle$ and the qubit is initially in the excited state $|1\rangle$. Use the previous results to show that:

$$\hat{H}_{\text{int}}\,|0,1\rangle = \hbar g\,|1,0\rangle$$
$$\hat{H}^2_{\text{int}}\,|0,1\rangle = (\hbar g)^2\,|0,1\rangle$$

(c) Because the interaction Hamiltonian is time-independent, the Schrödinger equation can be solved in the usual way to give the unitary evolution operator:

$$\hat{U}(t) = \exp\left(-\frac{it}{\hbar}\hat{H}_{\text{int}}\right) = \sum_{j=0}^{\infty}\left(-\frac{it}{\hbar}\right)^j \frac{\hat{H}^j_{\text{int}}}{j!}.$$

Given the initial state $|\psi(t)\rangle = |0,1\rangle$, use previous results and Taylor expansion of the unitary evolution operator to show that the state of the overall system after time t will be:

$$|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle = \cos(gt)\,|0,1\rangle - i\sin(gt)\,|1,0\rangle$$

[Hint: $\cos\theta = \sum_{j=0}^{\infty}\frac{(-1)^j\theta^{2j}}{(2j)!} \approx 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots$ and $\sin\theta = \sum_{j=0}^{\infty}\frac{(-1)^j\theta^{2j+1}}{(2j+1)!} \approx \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots.$]

(d) Calculate and plot the excited state probability as a function of time t, i.e. compute $|\langle 1,0|\psi(t)\rangle|^2$. Find the time in terms of g needed to reach the excited and 50-50 superposition states.
[Hint: Use the orthogonality relation $\langle 1,0|0,1\rangle = \langle 1|0\rangle_{\text{cavity}}\langle 0|1\rangle_{\text{qubit}} = 0$.]

Note: Even if you don't manage to solve parts of the problem, you can still use the results to work on the successive ones.

Question 6 *(Quantum gates)*

Consider the unitary operator:

$$\hat{U}_{\text{qq}}(t) = \exp\left[-i\frac{g}{2}(\hat{\sigma}_x\otimes\hat{\sigma}_x + \hat{\sigma}_y\otimes\hat{\sigma}_y)t\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(gt) & -i\sin(gt) & 0 \\ 0 & -i\sin(gt) & \cos(gt) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3)$$

(a) How is this unitary operator obtained and what physical process does this operator correspond to? What is the significance of this process in the context of quantum computing?

(b) Consider the following initial state at $t=0$:

$$|\psi(0)\rangle = a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle,$$

where a,b,c,d are complex probability amplitudes.
What is the state of the system after the unitary evolution at time t? Find t_E and initial probability amplitudes a,b,c,d to obtain the states:

$$|\psi(t_E)\rangle = \frac{1}{\sqrt{2}}(|01\rangle - i\,|10\rangle)$$
$$|\psi(t_E)\rangle = \frac{1}{\sqrt{2}}(|10\rangle - i\,|01\rangle).$$

What are the quantum properties of these two states? For this choice of $t = t_E$, what quantum gate operation does the unitary operator corresponds to?

We have the standard definitions: $|00\rangle = |0\rangle \otimes |0\rangle$, $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Pauli matrices: $\hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\hat{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$. Also $\hat{\mathbb{1}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the 2×2 identity matrix.

Tensor product definition: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} a \cdot \begin{bmatrix} x & y \\ z & w \end{bmatrix} & b \cdot \begin{bmatrix} x & y \\ z & w \end{bmatrix} \\ c \cdot \begin{bmatrix} x & y \\ z & w \end{bmatrix} & d \cdot \begin{bmatrix} x & y \\ z & w \end{bmatrix} \end{bmatrix}$.

Also, remember $|01\rangle = |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.