MS-C2105 Introduction to Optimization | Philine Schiewe

Exam

08.06.2023

- You need to justify all of your answers.
- You can use a non-graphing calculator.
- You can use one two-sided cheat sheet (maximal DIN A4).

Problem 1 (15 points)

Consider the following linear program

$$
\begin{aligned}
(P) \quad \max \quad & x_1 + 2x_2 \\
\text{s.t.} \quad & 3x_1 + x_2 \le 6 \\
& -x_1 + 2x_2 \le 4 \\
& x_2 \le 3 \\
& x_1, \quad x_2 \ge 0.
\end{aligned}
$$

a) (10 points) Give the standard form of (P) and compute an optimal solution using the simplex method.

b) (5 points) Formulate the dual (D) of (P) and give an optimal solution of (D). ***Hint:*** *You can use the information from a) to find an optimal solution of the dual (D).*

Problem 2 (15 points)

A group of friends is planning a vacation. They want to cover the following points: flights, hotel, airport transfer, rental bikes, breakfast and a day trip. Their goal is to minimise the costs of the vacation. They can choose between different packages with the following costs.

package	flights	hotel	airport transfer	rental bikes	breakfast	day trip	costs
A	✓	✓	✓		✓		2000€
B	✓			✓		✓	1100€
C		✓			✓		800€
D			✓	✓		✓	400€
E	✓		✓	✓		✓	1300€

a) (10 points) Model the problem as a binary integer program.

b) (5 points) Packages B and D are offered by the same company. If both packages are purchased, there is a 250€ discount. How do you have to change the model from a) to add this, such that the model remains a binary integer program?

Hint: *You do not need to solve this problem.*

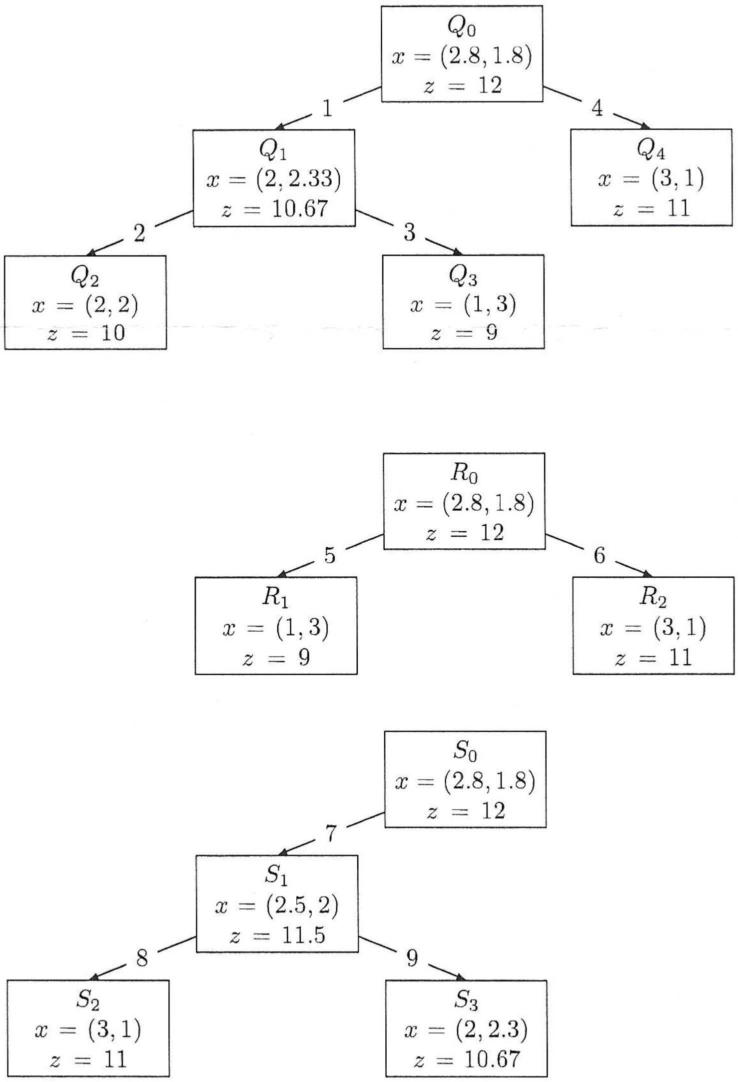
Problem 3 (15 points)

The following integer program (P) is solved by the branch-and-bound method.

$$
\begin{aligned}
(P) \quad & \max 3x_1 + 2x_2 \\
& \text{s.t. } 4x_1 + x_2 \leq 13 \\
& 2x_1 + 3x_2 \leq 11 \\
& x_1, x_2 \geq 0 \\
& x_1, x_2 \in \mathbb{Z}
\end{aligned}
$$

Consider the branch-and-bound trees Q, R and S in the figure below where the subproblems in each tree are solved in the order Q_0, Q_1, Q_2, Q_3, Q_4 for tree Q, R_0, R_1, R_2 for tree R and S_0, S_1, S_2, S_3 for tree S.

For each edge $1, \ldots, 9$ give a constraint that, when added to the formulation, results in the optimal LP solution of the corresponding branch. Argue for each tree whether it corresponds to a correct implementation of the branch-and-bound algorithm. What is an optimal solution of (P)?



Problem 4 (35 points)

Consider the unconstrained optimisation problem

$$\min \; f(x_1, x_2) = (3 - x_1)^2 - 4x_1x_2 + 8x_2^2.$$

a) (5 points) Compute an optimal solution x^* analytically and give the corresponding necessary and sufficient optimality conditions.

b) (15 points) Compute one step of the gradient descent method with $x_0 = (0,0)^\top$, $\varepsilon = 0.001$ and optimal stepwidth λ. Justify why the resulting solution is or is not optimal without relying on the solution x^* from a).

c) (15 points) Compute one step of Newton's method with $x_0 = (0,0)^\top$, $\varepsilon = 0.001$ and optimal stepwidth λ. Justify why the resulting solution is or is not optimal without relying on the solution x^* from a).

Hint: *The eigenvalues of* $A = \begin{pmatrix} 2 & -4 \\ -4 & 16 \end{pmatrix}$ *are approximately 17.062 and 0.938,* $A^{-1} = \begin{pmatrix} 1 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{8} \end{pmatrix}$.

Hint: *If you cannot compute the optimal stepwidth, you can use* $\lambda = 1$ *in your computations. Note that this will not result in full points.*

Problem 5 (10 points)

Consider the following optimization problem

$$\begin{aligned} (P) \quad & \min \; x_1x_2^2 - 5x_1 + (x_2 - 3)^2 \\ & \text{s.t. } 2x_1 - 3x_2 = 5 \\ & \qquad x_1^2 + 3x_2 \le 4 \end{aligned}$$

a) (4 points) Formulate the KKT conditions for (P).

b) (6 points) Formulate the barrier problem for (P) and the corresponding Newton system for the first order optimality condition.

Hint: *You do not have to find an optimal solution for* (P).

Problem 6 (10 points)

Consider the two problems (P) and (P').

$$\begin{aligned} (P) \quad & \max \; c^\top x \\ & \text{s.t. } Ax \le b \\ & \qquad x \ge 0 \\ & \qquad x \in \mathbb{Z}^n \end{aligned} \qquad\qquad \begin{aligned} (P') \quad & \max \; c^\top x \\ & \text{s.t. } Ax \le b \\ & \qquad x \ge 0 \end{aligned}$$

a) (3 points) Show that if (P) has a feasible solution, (P') also has a feasible solution.

b) (7 points) Show that the reverse is not true, i.e., there exist A, b, c such that (P) is infeasible while (P') is feasible.
Hint: *You can construct an example to show this.*