Foundations of discrete mathematics
MS-A0402
Department of mathematics and systems analysis
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Exam
19.4.2024
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Instructions: Answer in English. Write clearly and justify your answers. Allowed equipment: writing tools, and a handwritten cheat sheet of at most A4 size, both sides can be used. **No calculators.**

P1 Let p, q, r be any propositions. Prove or disprove, directly using a truth table of eight rows, that

$$(p \to q) \wedge (q \to r) \wedge (r \to p)$$

is true if and only if p, q and r all have the same truth value. **(6p)**

P2 Consider the sums

$$s_n = \sum_{k=1}^{n} k = 1 + 2 + \ldots + n$$
$$t_n = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2$$

where $n \geq 1$.

(a) Prove by induction that $s_n = n(n+1)/2$ for all $n \geq 1$. **(3p)**

(b) Calculate the numbers $t_1, t_2, \ldots, t_6$. **(1p)**

(c) Prove by induction that $t_n \geq n^3/6$ for all $n \geq 1$. **(2p)**

P3 Let $S = \{a, b, c, d, e, f, g, h, i, j\}$, and let $\pi : S \to S$ be the permutation

$$(a\,b\,c\,d)(e\,f\,g\,h\,i\,j),$$

written in disjoint cycle notation.

(a) Write π^2 in disjoint cycle notation. **(1p)**

(b) Write π^4 in disjoint cycle notation. **(1p)**

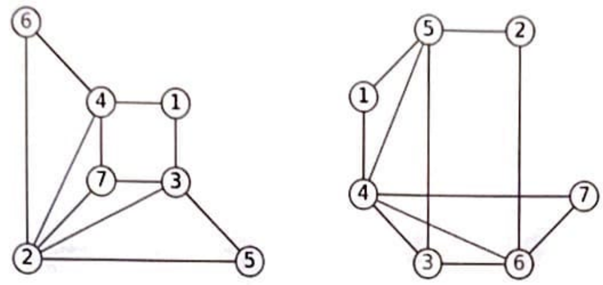
(c) Write π^{-1} in disjoint cycle notation. **(1p)**

(d) Give some positive integer n such that π^n is the identity permutation. Give a reason why it is the identity permutation. **(1p)**

(e) Find the *smallest* positive integer n such that π^n is the identity permutation. Give a reason why it is the smallest such integer. **(1p)**

(f) Write π^{100} in disjoint cycle notation. **(1p)**

P4 Consider the following two graphs (G on the left, H on the right).



(a) Are G and H isomorphic? If they are, give an isomorphism f from G to H, by listing the values $f(1), f(2), \ldots, f(7)$. If they are not isomorphic, explain why. **(3p)**

(b) Find the clique number of G (the size of the largest clique). **(0.5p)**

(c) Find the maximum degree of G (the largest degree that any vertex has). **(0.5p)**

(d) Based on the information from (b) and (c), what can be said about the number of colors needed to color G so that no two adjacent vertices have the same color? **(1p)**

(e) Find a coloring of G, using as few colors as possible, such that no two adjacent vertices have the same color. **(1p)**

P5

(a) Find $(5^k) \bmod 1000$ for $k \in \{1, 2, 3, 4, 5, 6\}$. **(2p)**

(b) Find $(5^{1000}) \bmod 1000$ by any method you like. **(2p)**

(c) Find $(9^{33}) \bmod 11$ by the method of repeated squaring. **(2p)**