Department of Mathematics and Systems Analysis
Aalto University, School of Science

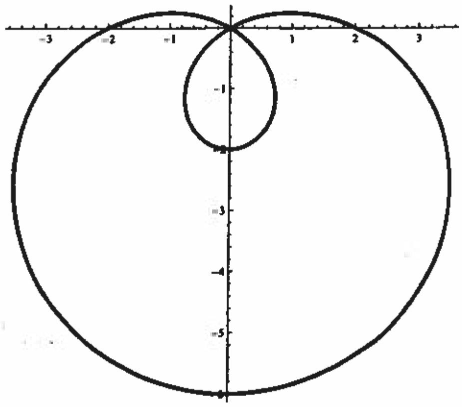
Metsalo

Mat-1.1620 Mathematics 2

Final Exam, 5 August 2013

Please fill in clearly *on every sheet* the data on you and the examination. On *Examination code* mark course code, title and text mid-term or final examination. Degree Programmes are ARK, AUT, BIO, EST, ENE, GMA, INF, KEM, KTA, KON, MAR, MTE, PUU, RRT, TFM, TIK, TLT, TUO, YYT.

Calculators are not allowed. The examination time is 4 hours. Ask, if You suspect typos in the text!



1. The curve in the figure to the right is a *limaçon*, given in polar coordinates as $r(\theta) = 2 - 4\sin\theta$. Calculate the area of the region inside the larger loop but outside the smaller loop of the limaçon.

2. Determine the maximum value of $T(x, y, z) = xy + yz$ on the unit sphere $x^2 + y^2 + z^2 = 1$ and the point/-s where T attains this maximum.

3. a) The linear, homogeneous differential equation (A): $y''(x) + 2y'(x) + y(x) = 0$ has constant coefficients, so it has a solution of the form $y_1(x) = e^{rx}$, if the constant r is chosen correctly. Determine r so that $y_1(x) = e^{rx}$ is a solution to (A). (1p.)
   b) (A) has also another, linearly independent solution $y_2(x)$, which can be obtained by setting $y_2(x) = c(x) \cdot y_1(x)$. Inserting this into (A) gives a 1^{st} order differential equation for $c'(x)$. Solve this differential equation and determine the general solution $C_1 y_1(x) + C_2 y_2(x)$ to (A). (2p.)
   c) The linear, inhomogeneous differential equation (B): $y''(x) + 2y'(x) + y(x) = x^2 + 1$ has a 2^{nd} degree polynomial as a particular solution. Determine this particular solution. (1p.)
   d) Determine the solution to (B), which satisfies the initial conditions $y(0) = 0, y'(0) = 0$. (2p.)
   (Note: It is easy to check a solution to a differential equation!)

4. If f and g are two functions of class $C^1(\mathbf{R})$, then $(fg)'(t) = f'(t)g(t) + f(t)g'(t)$.
   a) Let $\vec{u}(t) = u_1(t)\vec{\imath} + u_2(t)\vec{\jmath} + u_3(t)\vec{k}$ and $\vec{v}(t) = v_1(t)\vec{\imath} + v_2(t)\vec{\jmath} + v_3(t)\vec{k}$ be two 3-vector functions of class $C^1(\mathbf{R})$. Show that $(\vec{u} \bullet \vec{v})'(t) = \vec{u}'(t) \bullet \vec{v}(t) + \vec{u}(t) \bullet \vec{v}'(t)$.
   b) Show that if $\vec{F}$ and $\vec{G}$ are two vector fields of class $C^1(\mathbf{R}^3)$, then $\nabla \bullet (\vec{F} \times \vec{G}) = (\nabla \times \vec{F}) \bullet \vec{G} - \vec{F} \bullet (\nabla \times \vec{G})$, *i.e.* $div(\vec{F} \times \vec{G}) = (curl(\vec{F})) \bullet \vec{G} - \vec{F} \bullet (curl(\vec{G}))$.

5. Calculate the line integral $W = \int_C \vec{F} \bullet d\vec{r}$, where C is the straight line from the origin $(0, 0, 0)$ to the point $(1, 2, 3)$ and $\vec{F}(x, y, z) = (e^z + \frac{1}{1+x^2})\vec{\imath} + 2yz\vec{\jmath} + (xe^z + y^2)\vec{k}$
   a) directly as a line integral,
   b) by showing that $\vec{F}$ is conservative, determining a scalar potential $\Phi(x, y, z)$ such that $\nabla\Phi = \vec{F}$ and calculating W using Φ.

Useful (?) formulas:
$\cos^2 t + \sin^2 t = 1$, $\cos^2 t = (1 + \cos(2t))/2$, $\sin^2 t = (1 - \cos(2t))/2$,
$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \arctan \frac{x}{a} + C \ (a > 0)$.