Department of Mathematics and Systems Analysis
Aalto University, School of Science

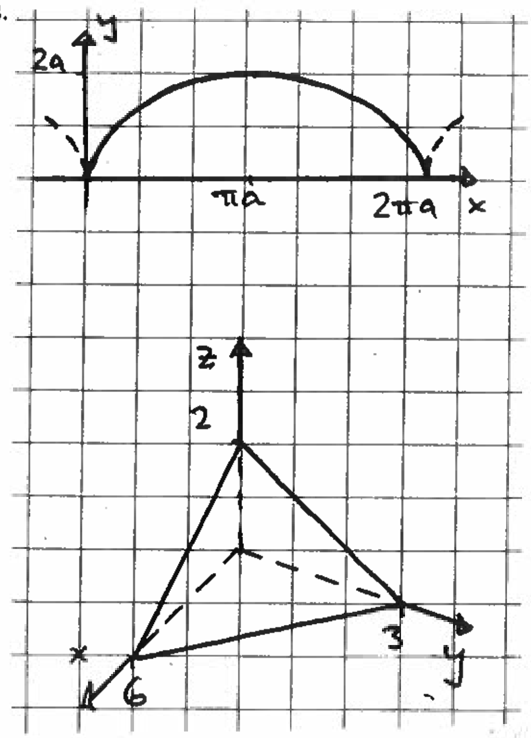
Metsalo

Mat-1.1620 Mathematics 2

Final Exam, 28 August 2013

Please fill in clearly *on every sheet* the data on you and the examination. On *Examination code* mark course code, title and text mid-term or final examination. Degree Programmes are ARK, AUT, BIO, EST, ENE, GMA, INF, KEM, KTA, KON, MAR, MTE, PUU, RRT, TFM, TIK, TLT, TUO, YYT.

Calculators are not allowed. The examination time is 4 hours. Ask, if You suspect typos in the text!



1. An arch of the cycloid can be given parametrically as $x(t) = a(t - \sin t), y(t) = a(1 - \cos t), 0 \leq t \leq 2\pi, a > 0$ (see the figure to the right). Its lenght is $8a$. If we start at the origin (corresponding to $t = 0$) and walk a distance $2a$ along the arch, where do we end up?

2. The point $P(-1, 1, -1)$ is on the ellipsoid $3x^2 + 2y^2 + z^2 = 6$. The normal line to the ellipsoid at P intersects the ellipsoid at another point Q as well. Determine Q.

3. Determine the maximum volume of a rectangular box, that fits into the tetrahedron in the figure to the right with vertices (0,0,0), (6,0,0), (0,3,0) and (0,0,2) so that three of its faces coincide with the coordinate planes.

4. $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}, \vec{x} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$

   a) Show that $\vec{x}$ is an eigenvector to A and determine the eigenvalue it corresponds to.
   b) 1 is an eigenvalue to A. Determine an eigenvector ($\neq \vec{0}$) corresponding to this eigenvalue.
   c) For a general square matrix B we have that $\det(B) = \det(B^T)$. Hence A and its transpose A^T have the same eigenvalues, since they have the same characteristic polynomial: $p_A(\lambda) = \det(A - \lambda I) = \det((A - \lambda I)^T) = \det(A^T - \lambda I) = p_{A^T}(\lambda)$. Determine an eigenvector ($\neq \vec{0}$) to A^T, corresponding to the eigenvalue 1.

5. $\vec{F}(x, y) = \frac{-y}{x^2+y^2}\vec{\imath} + \frac{x}{x^2+y^2}\vec{\jmath}$, where $(x, y) \in \mathbf{R}^2 \setminus \{(0, 0)\}$.

   a) Show that $\nabla \bullet \vec{F} \equiv 0$ in $\mathbf{R}^2 \setminus \{(0, 0)\}$.
   b) Show that $\nabla \times \vec{F} \equiv \vec{0}$ in $\mathbf{R}^2 \setminus \{(0, 0)\}$.
   c) Calculate $\oint_C \vec{F} \bullet d\vec{r}$, where C is the unit circle in the xy-plane, taken counterclockwise.

Useful (?) formulas:
$\cos^2 t + \sin^2 t = 1$, $\cos^2 t = (1 + \cos(2t))/2$, $\sin^2 t = (1 - \cos(2t))/2$,