Mat-1.1632 Mathematics 3-II

Please fill in the required information into each paper sheet. Calculators are allowed.

1. **(a)** State the fundamental theorem for linear systems. (2p)

(b) Solve the following linear system by the Gauss elimination: $\begin{cases} 4y+3z=8 \\ 2x-\ \ z=2 \\ 3x+2y=5 \end{cases}$. (4p)

2. Solve the linear system by Doolittle's method, showing the details, in particular the LU factorization.

$$\begin{cases} x_1+x_2+x_3=5 \\ x_1+2x_2+2x_3=6 \\ x_1+2x_2+3x_3=8 \end{cases}$$ (6p)

3. Find a general solution of the system $\begin{cases} \dot{y}_1=4y_2 \\ \dot{y}_2=4y_1-4 \end{cases}$ (6p)

4. Find critical points of the system and determine their type and stability

$$\begin{cases} \dot{x}=y^2-y \\ \dot{y}=x \end{cases}$$ (6p)

5. Consider the mixed problem for heat equation

(1) $\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}$,

(2) $u(x,0)=f(x)$

(3) $u'_x(0,t)=0,\ \ u'_x(1,t)=0$

(a) Substituting $u(x,t)=X(x)T(t)$, derive the ordinary differential equations $X''+\lambda^2 X=0$ and $T'+\lambda^2 T=0$, where $\lambda\in\mathbf{R}$ is a constant. (1p)

(b) Find the function $X(x)$ satisfying the boundary conditions derived from Eq.(3), and the constant λ. Show that the solution to the problem (1),(3) can be presented in the form

(4) $$u(x,t)=B_0+\sum_{k=1}^{\infty}B_k\exp(-\pi^2k^2t)\cos(\pi kx)$$ (2p)

(c) Then find the solution that satisfies (2) (Find coefficients B_k). (1p)

(d) Find the solution to the problem (1), (2), (3), if $f(x)=1+\cos 2\pi x$ (2p)

TURN OVER→

Appendix. Fourier series

Any periodic (with period 2L) piecewise continuous in the interval $-L \le x \le L$ function $f(x)$ can be represented by the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{\pi n x}{L} + b_n \sin\frac{\pi n x}{L} \right),$$

where $$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{\pi n x}{L}dx\,, \quad n=1,2...;$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{\pi n x}{L}dx, \quad n=1,2...$$