Mathematics
Aalto University
Hannukainen/Autio/Khuat

MS-E1651 - Numerical Matrix Computations

Exam 16.10.2024

Please fill in clearly *on every sheet* the data on you and the examination. On *Examination code* mark course code, title and text mid-term or final examination.

The exam time is three hours (unless otherwise agreed before the exam). No electronic calculators or materials are allowed.

Solve all problems. The grade is best of the two:

- Grade is based only on the exam.
- Grade is based on exercise points and exam points.

1. Let $A \in \mathbb{R}^{n\times n}$ be s.p.d., $\{\boldsymbol{p}_i\}_{i=1}^n$ an A-orthonormal basis of $\mathbb{R}^n$, and denote $V = span\{\boldsymbol{p}_1, \boldsymbol{p}_2\}$.

   (a) (2p) Show that

   $$P_A := \boldsymbol{p}_1\boldsymbol{p}_1^T A + \boldsymbol{p}_2\boldsymbol{p}_2^T A$$

   is an A-orthogonal projection to V. **Hint:** $I = \sum_{i=1}^n \boldsymbol{p}_i\boldsymbol{p}_i^T A$.

   (b) (2p) Let $\boldsymbol{x} \in \mathbb{R}^n$. Show that

   $$\|\boldsymbol{x} - P_A\boldsymbol{x}\|_A < \|\boldsymbol{x} - P_A\boldsymbol{x} + \boldsymbol{v}\|_A$$

   for any $\boldsymbol{v} \in V$, $\boldsymbol{v} \neq 0$.

   (c) (2p) Let $\boldsymbol{x} \in \mathbb{R}^n$ satisfy $A\boldsymbol{x} = \boldsymbol{p}_1$. Assume that a line search method with search directions $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ is used to approximately solve $\boldsymbol{x}$ starting from initial guess zero. Compute the final line search iterate.

2. (a) (3p) Let

   $$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 + \frac{1}{4} \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 5 \\ 5 + \frac{1}{4} \end{bmatrix}.$$

   Compute the Cholesky decomposition of A and use it to solve the linear system $A\boldsymbol{x} = \boldsymbol{b}$.

   (b) (3p) Let $A \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix such that

   $$A = \begin{bmatrix} 1 & \boldsymbol{a}_{21}^T \\ \boldsymbol{a}_{21} & A_{22} \end{bmatrix} \quad \text{where } \boldsymbol{a}_{21} \in \mathbb{R}^{n-1} \text{ and } A_{22} \in \mathbb{R}^{(n-1)\times(n-1)}.$$

   Find the lower triangular matrix $L \in \mathbb{R}^{n\times n}$ such that

   $$A = L \begin{bmatrix} 1 & 0 \\ 0 & A_{22} - \boldsymbol{a}_{21}\boldsymbol{a}_{21}^T \end{bmatrix} L^T.$$

3. Consider the linear system: Find $\boldsymbol{x} \in \mathbb{R}^2$ such that

$$A\boldsymbol{x} = \boldsymbol{b}, \text{ where } A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} 0 \\ b_2 \end{bmatrix}.$$

Assume that $a_{11}, a_{22} \neq 0$ and $\boldsymbol{x}$ is solved in floating point arithmetic using back-substitution. In addition, let $|\delta_1| < u$, $|\delta_2| < u$ and $u \ll 1$.

(a) (1p) Show that $(1+\delta_1)(1+\delta_2) = 1 + \theta_1$ for $|\theta_1| \leq 3u$.

(b) (1p) Show that $(1+\delta_1)^{-1} = 1 + \theta_2$ for $|\theta_2| \leq \frac{u}{1-u}$.

(c) (1p) Show that $(1+\delta_1)^{-1}(1+\delta_2)^{-1} = 1 + \theta_3$ for $|\theta_3| \leq 3\frac{u}{1-u}$.

(d) (3p) Find $\epsilon_{11}, \epsilon_{22} \in \mathbb{R}$ such that

$$(A + \Delta A) fl(\boldsymbol{x}) = \boldsymbol{b} \quad \text{where} \quad \Delta A = \begin{bmatrix} \epsilon_{11} a_{11} & 0 \\ 0 & \epsilon_{22} a_{22} \end{bmatrix}$$

and give estimates for $|\epsilon_{11}|$ and $|\epsilon_{22}|$. **Hint:** note that x_1 is obtained by solving $a_{11}x_1 + a_{12} fl(x_2) = 0$ in floating point arithmetic.

4. Let $A \in \mathbb{R}^{n\times n}$, $\boldsymbol{b} \in \mathbb{R}^n$ be s.p.d. and $J : \mathbb{R}^n \to \mathbb{R}$ be defined as

$$J(\boldsymbol{u}) := \frac{1}{2}\boldsymbol{u}^T A \boldsymbol{u} - \boldsymbol{b}^T \boldsymbol{u}.$$

In addition, let $\boldsymbol{x} \in \mathbb{R}^n$ satisfy $A\boldsymbol{x} = \boldsymbol{b}$.

(a) (2p) Show that $J(\boldsymbol{x}) < J(\boldsymbol{x} + \boldsymbol{v})$ for any $\boldsymbol{v} \in \mathbb{R}^n$, $\boldsymbol{v} \neq 0$.

(b) (2p) Show that $\|\boldsymbol{x} - \boldsymbol{y}\|_A^2 = 2(J(\boldsymbol{y}) - J(\boldsymbol{x}))$ for any $\boldsymbol{y} \in \mathbb{R}^n$. **Hint:** first show that $J(\boldsymbol{x}) = -\frac{1}{2}\|\boldsymbol{x}\|_A^2$.

(c) (2p) Let $V \subset \mathbb{R}^n$ be a subspace and $\boldsymbol{y} \in V$ satisfy

$$\|\boldsymbol{x} - \boldsymbol{y}\|_A < \|\boldsymbol{x} - \boldsymbol{y} + \boldsymbol{v}\|_A$$

for any $\boldsymbol{v} \in V$, $\boldsymbol{v} \neq 0$. Use (b) to show that $J(\boldsymbol{y}) < J(\boldsymbol{y} + \boldsymbol{v})$ for any $\boldsymbol{v} \in V$, $\boldsymbol{v} \neq 0$.

Recall the formula $(a+b)^2 = a^2 + 2ab + b^2$