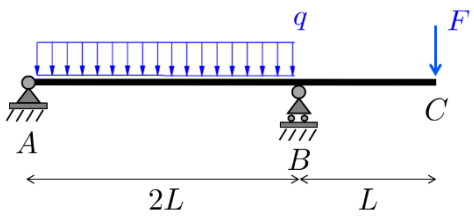
1. [1p] For a plane beam, define the internal forces in terms of the stresses. Explain the meaning of the corresponding stress resultants.

2. [2p] Explain the main difference between the Bernoulli and Timoshenko beam models. For both beam models, write down a constitutive relation for the shear force, i.e., define the shear force in terms of the corresponding kinematical variables. Name all the quantities and variables.

3. [1p] Explain the main principle of the force method. List the basic steps of solving a problem using the force method.

4. [1p] In the fundamental slope-deflection equation $M_{ij} = a_{ij}\varphi_{ij} + b_{ij}\varphi_{ji} - c_{ij}\psi_{ij} + \bar{M}_{ij}$, name all the quantities. What is the physical meaning of coefficients a_{ij}, b_{ij} and c_{ij}? Briefly explain how to determine these coefficients.

5. [5p] Given a beam of length $3L$ with a simple fixed support in *A* and a simple support permitting horizontal motion in *B*. The span *AB* has length $2L$. The span *BC* has length L. The beam is linear elastic with constant bending stiffness EI, constant shear stiffness GA_z, and constant extension stiffness EA. The beam is loaded by a uniformly distributed load with intensity q acting on the span *AB* and a concentrated force F acting at the tip in *C*.

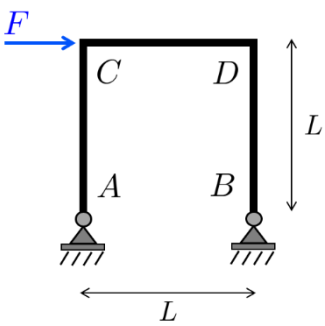


a) Determine the support reactions. Hint: Use balance equations for that.

b) Assuming that $F = \frac{qL}{3}$, determine and draw the distributions of the bending moment, shear and axial forces.

c) Use the principle of virtual work to determine the vertical displacement w_C at the tip *C*.
In the equation of virtual work, account for effects of bending moment, shear and axial forces.

6. [5p] Given a rectangular frame with simple supports in *A* and *B*. A horizontal force F is acting at the joint *C*. The beams *AC*, *CD,* and *DB* have length L. The beams are linear elastic with constant bending stiffness EI.

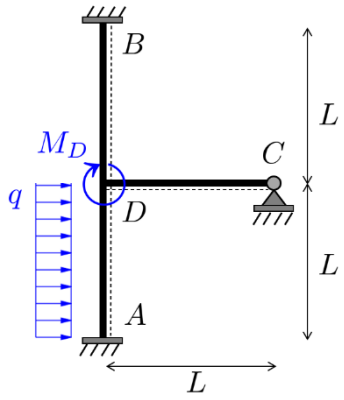


a) Use the force method to determine the internal moment in the joint *C*.
Account only for effect of bending moment.

b) Determine the reactions.

c) Determine and draw the distributions of the bending moment, shear and axial forces.

7. [5p] Given a T-frame with fixed supports in *A* and *B* and a simple support in *C*. The beams *AD*, *DC,* and *DB* have length L. A uniformly distributed load with intensity q is acting in the horizontal direction on the beam *AD*. A concentrated moment M_D is applied at joint *D* in the clockwise direction as shown in the figure. The beams are linear elastic with constant bending stiffness EI.



a) Use the Slope-Deflection Method and determine the **bending moment** at fixed support *A* and **rotation angle** at node *C*.

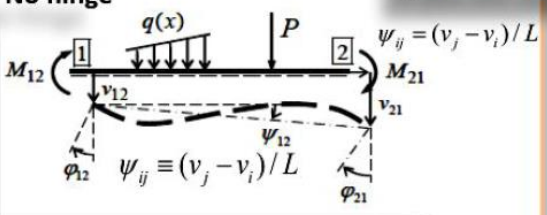
b) Assuming that $M_D = qL^2$, determine and draw the bending moment distribution. Remember the sign convention for the bending moment.

Supporting materials: The Slope-Deflection Method.

The stiffness equation relating the end-moments to the end-displacements

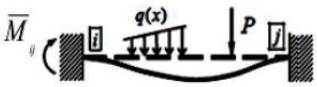
If you are using lecture's notations

No hinge



$$M_{ij} = a_{ij}\varphi_{ij} + b_{ij}\varphi_{ji} - c_{ij}\psi_{ij} + \bar{M}_{ij},\ i \neq j$$

$$a_{ij} = \frac{4EI}{L},\ b_{ij} = \frac{2EI}{L},\ c_{ij} = \frac{6EI}{L}$$ (*EI*-constant)



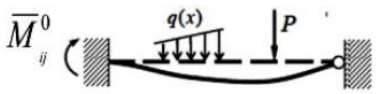
Fixed end-moment resulting from external mechanical loading, look from tables

One node is hinged

The is a superscript "0" means that the support at end *j* is hinged

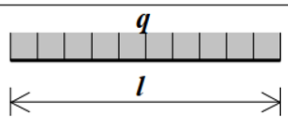
$$M_{ij}^{0} = a_{ij}^{0}\varphi_{ij} - c_{ij}^{0}\psi_{ij} + \bar{M}_{ij}^{0}$$

$$a_{12}^{0} = c_{12}^{0} = \frac{3EI}{L} \qquad \psi_{ij} = (v_j - v_i)/L$$

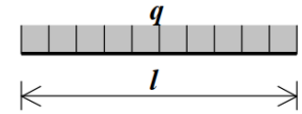


Fixed end-moment resulting from external mechanical loading, look from tables

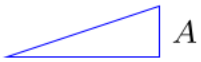
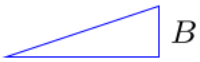
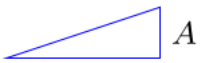
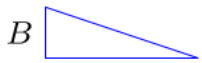
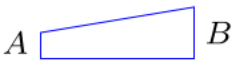
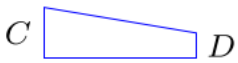
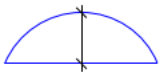
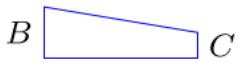
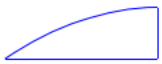
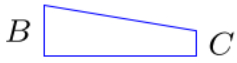
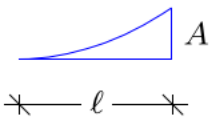
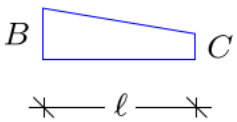
Fixed end moments for fixed member.

Loading	$\bar{M}_{ij}$ $l = l_{ij}$ $\bar{M}_{ji}$
1 	$\bar{M}_{ij} = -\frac{ql^2}{12},\quad \bar{M}_{ji} = \frac{ql^2}{12}$

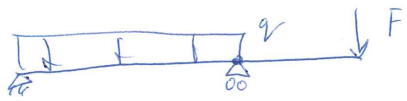
Fixed end moments for member having pin support at one end.

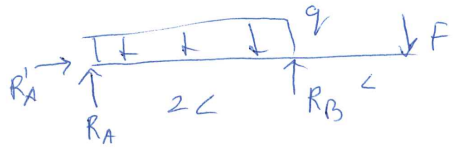
Loading	$\bar{M}_{ij}$ $l = l_{ij}$	$\bar{M}_{ji}$ $l = l_{ij}$
1 	$\bar{M}_{ij} = -\frac{ql^2}{8}$	$\bar{M}_{ji} = \frac{ql^2}{8}$

Supporting materials: Product integrals for simple shapes.

$\int_0^\ell f(x)\,g(x)\,dx$	$f(x)$	$g(x)$
$\frac{\ell}{3}\,A\,B$		
$\frac{\ell}{6}\,A\,B$		
$\frac{\ell}{6}\left(2AC+2BD+AD+BC\right)$		
$\frac{\ell}{3}\,A\left(B+C\right)$		
$\frac{\ell}{12}\,A\left(3B+5C\right)$		
$\frac{\ell}{12}\,A\left(B+3C\right)$		

Ex. 5





a). $R_A' = 0$

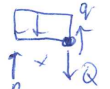
$A\circlearrowleft: \; -q\cdot 2\ell^2 + R_B \cdot 2\ell - F\cdot 3\ell = 0 \Rightarrow R_B = \frac{3}{2}F + q\ell$

$R_A + R_B - q\cdot 2\ell - F = 0 \Rightarrow R_A = F + 2\ell q - \frac{3}{2}F - q\ell \Rightarrow R_A = -\frac{F}{2} + q\ell$

---

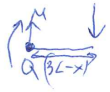
b). $F = \frac{q\ell}{3} \Rightarrow \quad R_B = \frac{3}{2}\cdot\frac{q\ell}{3} + q\ell = \frac{3}{2}q\ell$

$R_A = -\frac{1}{2}\cdot\frac{q\ell}{3} + q\ell = \frac{5}{6}q\ell$



$R_A - Q - qx = 0 \Rightarrow Q = q\left(\frac{5}{6}\ell - x\right)$

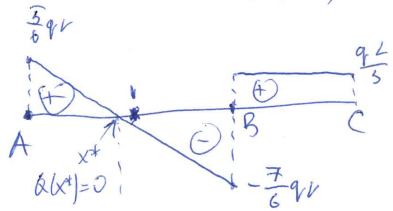
$M + q\frac{x^2}{2} - R_A x = 0 \Rightarrow M = \frac{q}{2}x\left(\frac{5}{3}\ell - x\right)$



$Q = \frac{q\ell}{3}$

$-M - \frac{q\ell}{3}(3\ell - x) = 0 \Rightarrow M = \frac{q\ell}{3}(x - 3\ell)$

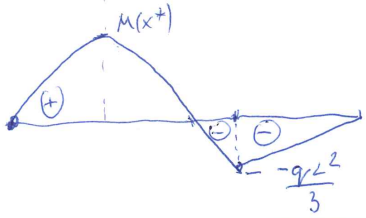
M:
Q:



$Q(0) = \frac{5}{6}q\ell$

$Q(2\ell) = -\frac{7}{6}q\ell$

$Q = \frac{q\ell}{3}$

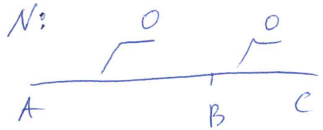


$M(0) = 0$

$M(2\ell) = -\frac{q\ell^2}{3}$

$M(2\ell) = -\frac{q\ell^2}{3}$

$M(3\ell) = 0$



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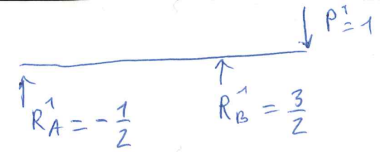
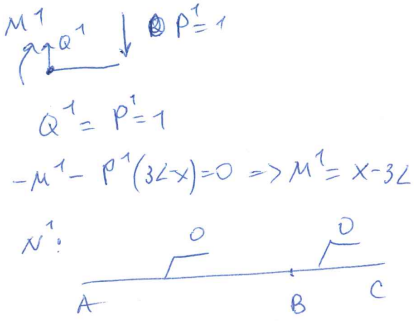
c). $P\cdot w_C = \int_{ABC}\left(\frac{N^0 N^1}{EA} + \frac{Q^0 Q^1}{GA_z} + \frac{M^0 M^1}{EJ}\right)ds$

Ex. 5

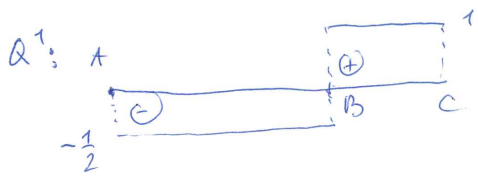


$Q^1 = R_A^1 = -\frac{1}{2}$

$M^1 - R_A^1 x = 0 \Rightarrow M^1 = -\frac{x}{2}$



Q^1:



M^1:

$M(0) = 0$

$M(2\ell) = -\ell$

$M(3\ell) = 0$

$$\neq \int_{ABC} \frac{Q^0 Q^1}{GA_2} ds = \int_0^{2\ell} Q^0 Q^1 dx + \int_{2\ell}^{3\ell} Q^0 Q^1 dx =$$

$$= \frac{2\ell}{6}\left(2\cdot\frac{5}{6}q\ell\left(-\frac{1}{2}\right) + 2\cdot\left(-\frac{7}{6}q\ell\right)\left(-\frac{1}{2}\right) + \frac{5}{6}q\ell\cdot\left(-\frac{1}{2}\right) + \left(-\frac{7}{6}q\ell\right)\cdot\left(-\frac{1}{2}\right)\right) + \ell\cdot 1\cdot\frac{q\ell}{3}$$

$$= -\frac{\ell}{6}\cdot q\ell\left(\frac{10}{6} - \frac{14}{6} + \frac{5}{6} - \frac{7}{6}\right) + \frac{q\ell^2}{3} = \frac{q\ell^2}{6} + \frac{q\ell^2}{3} = \frac{q\ell^2}{2}$$

$$\neq \int_{ABC} \frac{M^0 M^1}{GA_2} ds = \int_0^{2\ell} (\cdot)\cdot 1\, dx + \int_{2\ell}^{3\ell} (\cdot)\, dx = -\frac{q\ell^4}{9} + \frac{q\ell^4}{9} = 0$$

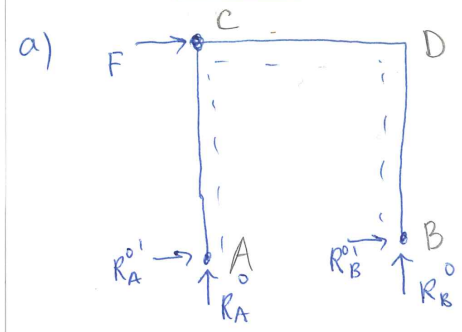
$$W_C = \int_{ABC} \frac{Q^0 Q^1}{GA_2} ds = \frac{q\ell^2}{2GA_2}$$

30 min

Ex. 6

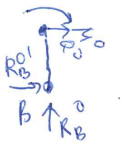
Case 0:

a)



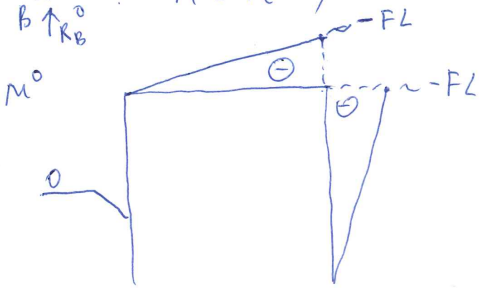
$R_A^{0'} = 0, \quad R_B^{0'} = -F$

$R_B^0 = F, \quad R_A^0 = -F$

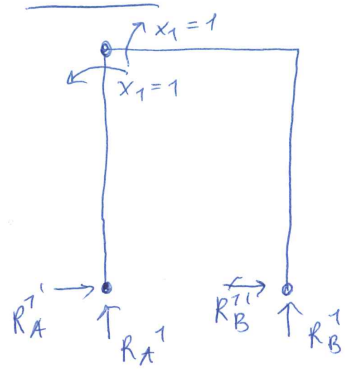


$-M^0 + R_B^{0'}(L-x) = 0$

$M^0 = F(x-L)$

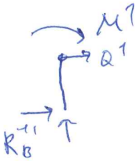


Case 1:



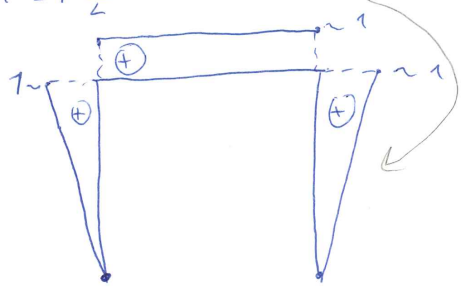
$R_A^{1'} = -\frac{1}{L}, \quad R_B^{1'} = \frac{1}{L}$

$R_B^1 = 0, \quad R_A^1 = 0$



$-M^1 + R_B^{1'}(L-x) = 0$

$M^1 = 1 - \frac{x}{L}$



$$\delta_{10} = \int_{AB} \frac{M^1 M^0}{EJ} ds = \frac{1}{EJ}\left(-\frac{FL^2}{2} + \frac{L}{3}(-FL)\cdot 1\right) = -\frac{5FL^2}{6EJ}$$

$$\delta_{11} = \int_{AB} \frac{M^1 M^1}{EJ} ds = \frac{1}{EJ}\left(2\frac{L}{3} + L\right) = \frac{5L}{3EJ}$$

$$\delta_1 = \delta_{10} + \delta_{11} X_1 = 0 \Rightarrow X_1 = -\frac{\delta_{10}}{\delta_{11}} = \frac{5FL^2 \cdot 3EJ}{6EJ \cdot 5L \cdot 2} = \boxed{\frac{FL}{2} = M_C}$$

b) $R_A^{0'} = 0 + \frac{FL}{2}\left(-\frac{1}{L}\right) = -\frac{F}{2}$

$R_B^{1'} = -F + \frac{FL}{2}\cdot\frac{1}{L} = -\frac{F}{2}$

$R_A = -F + \frac{FL}{2}\cdot 0 = -F$

$R_B = F + \frac{FL}{2}\cdot 0 = F$

Ex. 6)

c). AC: $N = F$

$M = \frac{F \cdot x}{2}$

$Q = \frac{F}{2}$

CD: $N = -\frac{F}{2}$

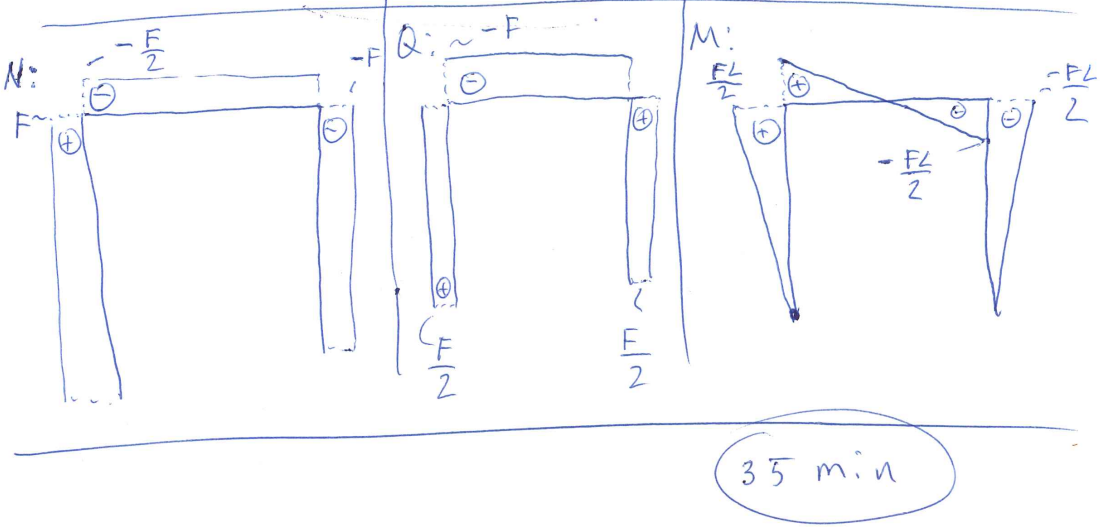
$M = \frac{F}{2}(\ell - 2x)$

$Q = -F$

DB: $N = -F$

$M = -\frac{F}{2}(\ell - x)$

$Q = \frac{F}{2}$



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Ex. 7)

a) $\varphi_{BD} = \varphi_{AD} = 0$

$\varphi_{DB} = \varphi_{DA} = \varphi_{DC} = \varphi_D$

$\varphi_{CD} = \varphi_C$

no sway

$n_k = 2$

$M_{DB} + M^{\circ}_{DC} + M_{DA} = M_D$

$M_{DB} = a_{DB}\varphi_D$

$M^{\circ}_{DC} = a^{\circ}_{DC}\varphi_D$

$M_{DA} = a_{DA}\varphi_D + \bar{M}_{DA}$

$a_{DB} = a_{DA} = \frac{4EJ}{L}$

$a^{\circ}_{DC} = \frac{3EJ}{L}$

$\bar{M}_{DA} = \frac{qL^2}{12}$

$\bar{M}_{AD} = -\frac{qL^2}{12}$

$b_{AD} = \frac{2EJ}{L}$

$a_{CD} = \frac{4EJ}{L}$

$b_{CD} = \frac{2EJ}{L}$

$\varphi_D\left(\frac{4EJ}{L}\cdot 2 + \frac{3EJ}{L}\right) + \frac{qL^2}{12} = M_D$

$11\frac{EJ}{L}\cdot\varphi_D = M_D - \frac{qL^2}{12} \Rightarrow \varphi_D = \left(M_D - \frac{qL^2}{12}\right)\cdot\frac{L}{11EJ}$

$M_{AD} = b_{AD}\varphi_D + \bar{M}_{AD} = \frac{2EJ}{L}\left(M_D - \frac{qL^2}{12}\right)\cdot\frac{L}{11EJ} - \frac{qL^2}{12} = \frac{2}{11}M_D - \frac{qL^2}{12}\left(1+\frac{2}{11}\right)$

$M_{AD} = \frac{2}{11}M_D - \frac{13}{11}\cdot\frac{qL^2}{12}$

$M_{CD} = 0 \Rightarrow a_{CD}\varphi_C + b_{CD}\varphi_D = 0 \Rightarrow \varphi_C = -\frac{b_{CD}}{a_{CD}}\varphi_D = -\frac{1}{2}\varphi_D = -\left(M_D - \frac{qL^2}{12}\right)\frac{L}{22EJ}$

---

b) $\exists\, M_D = qL^2 \Rightarrow M_D - \frac{qL^2}{12} = \frac{11}{12}qL^2$

$\varphi_D = \frac{qL^3}{12EJ}$

$M_{DB} = \frac{4EJ}{L}\cdot\frac{qL^3}{12EJ} = \frac{qL^2}{3}$

$M^{\circ}_{DC} = \frac{qL^2}{4}$

$M_{DA} = \frac{qL^2}{3} + \frac{qL^2}{12} = \frac{5qL^2}{12}$

$M_{CD} = 0$

$M_{AD} = \frac{2}{11}\cdot\frac{11}{12}qL^2 - \frac{qL^2}{12} = \frac{qL^2}{12}$

$M_{BD} = b_{BD}\varphi_D = \frac{qL^2}{6}$

Ex 7)

$M(x) = ax + b$

$M(0) = b = M_{DB}$

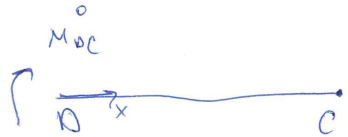
$M(L) = aL + M_{DB} = -M_{BD}$

$a = -\dfrac{M_{BD} + M_{DB}}{L}$

$\Rightarrow M(x) = -\left(M_{BD} + M_{DB}\right)\dfrac{x}{L} + M_{DB}$

$= -\left(\dfrac{qL^2}{6} + \dfrac{qL^2}{3}\right)\dfrac{x}{L} + \dfrac{qL^2}{3}$

$= \dfrac{qL^2}{3} - \dfrac{qL^2}{2}\cdot\dfrac{x}{L}$



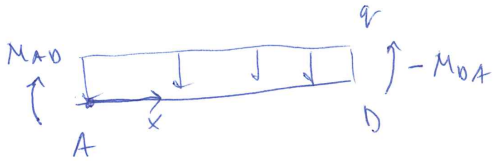
$M(x) = ax + b$

$M(0) = b = M_{DC}$

$M(L) = aL + b = 0 \Rightarrow a = -\dfrac{b}{L}$

$\rightarrow M(x) = -b\dfrac{x}{L} + b = b\left(1 - \dfrac{x}{L}\right)$

$= \dfrac{qL^2}{4}\left(1 - \dfrac{x}{L}\right)$



$M'' = -q$

$M(x) = -\dfrac{qx^2}{2} + ax + b$

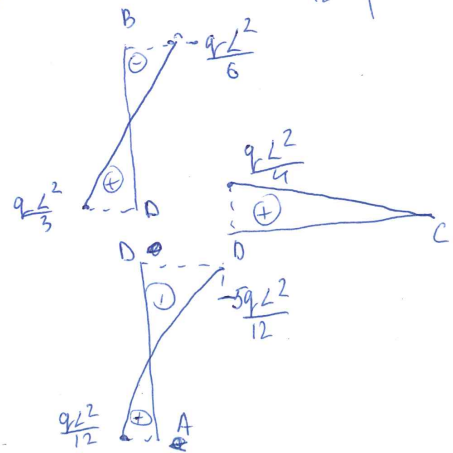
$M(0) = b = M_{AD} = \dfrac{qL^2}{12}$

$M(L) = -\dfrac{qL^2}{2} + aL + b = -M_{DA} = -\dfrac{5}{12}qL^2$

$-\dfrac{qL}{2} + a + \dfrac{qL}{12} + \dfrac{5}{12}qL = 0$

$a = qL\dfrac{6 - 1 - 5}{12} = 0$

$M(x) = -\dfrac{qx^2}{2} + \dfrac{qL^2}{12}$



28 min