# PHYS-E0410 Quantum Mechanics 2

Retake exam, 19.2.2025

- Please write your **name**, **student number**, **study program**, the **date**, and the **course code** PHYS-E0410 in all of your papers.
- **Please answer the different problems on separate sheets.** You do not have to do them in the order they appear in the problem set.
- You should answer in English unless you have special permission to use another language.
- Calculators, books, lecture notes, phones, computers, internet connections are forbidden. **Exception: you can bring 1 sheet (A4) on both sides of which you can have written anything you want prior to the exam.**
- This exam set consists of 4 problems. It is 4 pages long. The exam's duration is 3h.

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## Problem 1: A few questions

Answer briefly. **No explanations of your answers are required in this problem.**

**a)** What are the eigenvalues of an operator $\hat{P}$ such that $\hat{P}^5$ is the identity?

**b)** A coherent state is an eigenstate of the annihilation operator $\hat{a}$. Is it an eigenstate of the creation operator $\hat{a}^\dagger$?

**c)** Show that two fermions cannot be in the same quantum state.

**d)** The CHSH inequality concerns the correlations $C(\alpha, \beta)$ of measurement results of two spin 1/2 denoted $A$ and $B$, where spin $A$ is measured along axis $\alpha$ and spin $B$ is measured along axis $\beta$. The inequality reads:

$$|C(\alpha,\beta) + C(\alpha',\beta) + C(\alpha,\beta') - C(\alpha',\beta')| \leq 2. \tag{1}$$

Measuring the left-hand-side quantity $S = |C(\alpha,\beta) + C(\alpha',\beta) + C(\alpha,\beta') - C(\alpha',\beta')|$ for a particular two-spin state, researchers found $S = 2.697 \pm 0.015$. What did they conclude from this?

## Problem 2: Density matrix for a beam of electrons

We study an electron beam (made of multiple electrons). We are interested in the spin state.

**a)** The generic density matrix of a two-state system can be written $\hat{\rho} = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$,

where $a$, $b$ and $c$ are some complex coefficients. Show that $a = 1 - c$.

This beam has an isotropic distribution of spin: $\langle \hat{S}_x \rangle = \langle \hat{S}_y \rangle = \langle \hat{S}_z \rangle = 0$. All electrons are in the same state (described by the same ket or the same density matrix). We remind a representation of the spin operators:

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

**b)** Show that the isotropic distribution above is impossible if the electrons are in a pure state.

**c)** Find the density matrix in the eigenbasis of $\hat{S}_z$.

# Problem 3: Atomic clock

We study an atomic clock based on a cloud of Caesium atoms ${}^{133}_{55}\mathrm{Cs}$. The clock makes use of a transition between hyperfine levels of the Cs atom, that is, levels whose energy degeneracy is lifted by a coupling between the angular momentum of electrons of the atom (spin+orbital angular momentum) and the spin of the nucleus.

This problem is split in **two independent parts**. Most of the questions within each part can also be answered independently.

## Part 1: Level structure

**a) Electronic spin:** The electronic configuration of Cs is [...]$6s^1$: the only partially filled orbital is 6s, which is also the highest-energy occupied orbital.

Explain why the total spin of a pair of electrons occupying one (fully-occupied) orbital is 0 (Reminder: an example of orbital is $3p_{-1}$, which is characterized by quantum numbers $n = 3$, $\ell = 1$ and $m = -1$). Then explain why the total electronic spin of Cs is $1/2$.

**b) Electronic orbital angular momentum:** Explain why the *projection* of the orbital angular momentum of all the electrons of a closed (= fully-occupied) level is 0 (an example of level is 3p which includes orbitals $3p_{-1}$, $3p_0$ and $3p_1$).

**c)** In fact, the *total* orbital angular momentum of a closed level is also 0 (admitted here). Explain why the total orbital momentum of (all) electrons of Cs is 0 in the ground state.

**d)** $\hat{\vec{J}}$ is the total angular momentum of electrons: $\hat{\vec{J}} = \hat{\vec{S}} + \hat{\vec{L}}$ where $\hat{\vec{S}}$ is the spin of electrons and $\hat{\vec{L}}$ is the angular momentum of electrons. Because of a relativistic effect called spin-orbit coupling, the energy-eigenstates of Cs are eigenstates of $\hat{J}^2$.

Show that orbital 6s has a total electronic angular momentum characterized by $j = 1/2$.

**e)** The magnetic coupling between nuclear and electronic spin is associated to a Hamiltonian term $\hat{H}_H = A\,\hat{\vec{I}}\cdot\hat{\vec{J}}$ where $A$ is some coefficient and $\hat{\vec{I}}$ is the nuclear spin. The nuclear spin of ${}^{133}\mathrm{Cs}$ is $i = 7/2$ (meaning that the eigenvalue of $\hat{I}^2$ is $\hbar^2 i(i+1)$ with $i = 7/2$).

Show that the eigenstates of this Hamiltonian are eigenstates of operator $\hat{F}^2$ where $\hat{\vec{F}} = \hat{\vec{I}} + \hat{\vec{L}} + \hat{\vec{S}} = \hat{\vec{I}} + \hat{\vec{J}}$.

**f)** The eigenvalues of operator $\hat{F}^2$ are denoted $\hbar^2 f(f+1)$ with $f$ some real number. Show that the ground state of the ${}^{133}\mathrm{Cs}$ atom is split into two energy levels of total angular momentum $f = 3$ and $f = 4$. What is the degeneracy of each of these levels?

## Part 2: Principle of the atomic clock

The transition used in an atomic clock is the transition between states $|f = 3, m_f = 0\rangle$ and $|f = 4, m_f = 0\rangle$. These two levels are now respectively denoted $|f = 3, m_f = 0\rangle = |-\rangle$ and $|f = 4, m_f = 0\rangle = |+\rangle$, with energies $E_\pm = E_0 \pm \hbar\omega_0/2$ where $\hbar$ is the reduced Planck's constant. The frequency spacing $\omega_0$ is the current time standard.The point of a Cs atomic clock is to find out this frequency $\omega_0$ by tuning a microwave oscillator's frequency to be exactly $\omega_0$. In this second part, we study how this is done, that is, how one figures out when the frequency of the oscillator is equal to $\omega_0$.

In basis $\{|-\rangle, |+\rangle\}$, fixing the origin of energies at $E_0$, the Hamiltonian of a Cs atom is written

$$\hat{H} = \frac{\hbar\omega_0}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3}$$

In the atomic clock, a cloud of $^{133}$Cs atoms is pushed into a microwave cavity which resonates at frequency $\omega$, close to $\omega_0$. This cavity is the microwave oscillator mentioned above, and its resonance frequency $\omega$ is assumed to be tuneable. The cavity is driven with a resonant electromagnetic field, and the atoms interact with this field. We will assume that the coupling of one atom to the field inside the cavity results in the following additional Hamiltonian in basis $\{|-\rangle, |+\rangle\}$:

$$\hat{V}(t) = \hbar\omega_1 \cos(\omega t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

While the atomic cloud is in the cavity, the Hamiltonian of an atom of the cloud is $\hat{H} + \hat{V}(t)$. The state of an atom is described by $|\psi(t)\rangle = a_-(t)|-\rangle + a_+(t)|+\rangle$.

**g)** Write the differential equation governing $a_-(t)$ and $a_+(t)$.

**h)** To solve this system, we introduce the following change of variable: $a_\pm(t) = \gamma_\pm(t)e^{\mp i\omega t/2}$. Write the differential equation governing $\gamma_+(t)$ and $\gamma_-(t)$.

**i)** We will assume that the terms oscillating at high frequency ($\omega$ or $\omega_0$ or higher) can be neglected (this is known as the rotating-wave approximation). Show that the simplified equation governing $\gamma_\pm(t)$ is

$$\frac{d\gamma_\pm}{dt} = \pm\frac{i(\omega - \omega_0)}{2}\gamma_\pm - \frac{i\omega_1}{2}\gamma_\mp. \tag{5}$$

**j)** We now assume that $|\omega - \omega_0| \ll \omega_1$, so that the differential equation can be further simplified. Solve the simplified differential system to show that

$$\begin{cases} \gamma_-(t) &= \gamma_-(0)\cos\frac{\omega_1 t}{2} - i\gamma_+(0)\sin\frac{\omega_1 t}{2} \\ \gamma_+(t) &= \gamma_+(0)\cos\frac{\omega_1 t}{2} - i\gamma_-(0)\sin\frac{\omega_1 t}{2}, \end{cases} \tag{6}$$

where $\gamma_\pm(0)$ are the projections of the state at time $t = 0$.

**k)** The cloud of atoms is first prepared in the ground state $|-\rangle$ at time $t = 0$, and immediately pushed into the microwave cavity, where it spends a time $\tau = \frac{\pi}{2\omega_1}$. What is the atoms' state $\begin{pmatrix} a_-(\tau) \\ a_+(\tau) \end{pmatrix}$ when they leave the cavity?

After exiting the cavity, the atomic cloud travels for a time $T$ in a space where there are no electromagnetic fields, that is, where the Hamiltonian is simply $\hat{H}$.

**l)** What is the state $\begin{pmatrix} a_-(\tau+T) \\ a_+(\tau+T) \end{pmatrix}$ of an atom of the cloud after this time $T$?

**m)** Show that $\gamma_-(\tau + T) = \frac{1}{\sqrt{2}}e^{i(\omega-\omega_0)T/2}$ and $\gamma_-(\tau + T) = \frac{-i}{\sqrt{2}}e^{-i(\omega-\omega_0)T/2}$ (these coefficients describe the state at time $t + T$ in the rotating frame).

After this time $T$, the cloud enters a second cavity identical to the first one where it is again exposed to electromagnetic fields at frequency $\omega$. It spends the same time $\tau$ in this second cavity as in the first.

**n)** Find $\gamma_+(T+2\tau)$ and show that the probability for an atom of the cloud to be in state $|e\rangle$ at the time when it exits the second cavity (= the probability that the atom has transited from $|g\rangle$ to $|e\rangle$ during the whole trip $\tau + T + \tau$) is the following oscillating function:

$$P_{|g\rangle\to|e\rangle} = \frac{1}{2}\Big(1+\cos\big((\omega-\omega_0)T/2\big)\Big). \tag{7}$$

These oscillations are called Ramsey fringes.

**o)** One can tune the resonance frequency $\omega$ of the oscillator, and measure the probability of the transition by measuring the number of atoms that have transited from $|g\rangle$ to $|e\rangle$. Assuming that $\omega$ is already somewhat close to $\omega_0$ to begin with ($|\omega-\omega_0| < 1/T$), explain how one can determine when the equality $\omega = \omega_0$ is achieved.

Some help with trigonometry:

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|---|---|---|---|---|---|
| $\cos\pi/2$ | $0$ | $\sin\pi/2$ | $1$ | $\cos^2 a$ | $(1+\cos(2a))/2$ |
| $\cos\pi/3$ | $1/2$ | $\sin\pi/3$ | $\sqrt{3}/2$ | $\sin^2 a$ | $(1-\cos(2a))/2$ |
| $\cos\pi/4$ | $1/\sqrt{2}$ | $\sin\pi/4$ | $1/\sqrt{2}$ | | |
| $\cos\pi/6$ | $\sqrt{3}/2$ | $\sin\pi/6$ | $1/2$ | | |

**End of exam set**