ELEC-A7200 Signals and Systems

Exam

Problem 1. Consider the periodic signal

$$x(t) = 2\cos\left(\frac{2\pi t}{T_0}\right) - \cos\left(\frac{4\pi t}{T_0}\right)$$

with period $T_0 > 0$.

(a) Determine the average power of the signal:

$$P = \frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt$$

(1 p)

(b) Determine the coefficients x_k for the exponential Fourier series representation:

$$x(t) = \sum_{k=-\infty}^{\infty} x_k e^{j\frac{2\pi k}{T_0}t}, \quad x_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-j\frac{2\pi k}{T_0}t}\, dt$$

(2 p)

(c) Sketch the *two-sided amplitude spectrum.* (1 p)

(d) Sketch the *two-sided phase spectrum.* (1 p)

(e) Sketch the *one-sided power spectrum.* (1 p)

Problem 2. Consider the trapezoidal pulse

$$x_1(t) = \begin{cases} \frac{4t}{T}, & 0 \le t < \frac{T}{4}, \\ 1, & \frac{T}{4} \le t \le \frac{3T}{4}, \\ \frac{4(T-t)}{T}, & \frac{3T}{4} < t \le T, \\ 0, & \text{otherwise}, \end{cases}$$

with $T > 0$.

(a) Sketch $x_1(t)$ for $0 \le t \le T$. (1 p)

(b) Determine the time derivative

$$x_2(t) = \frac{d}{dt}x_1(t)$$

and sketch $x_2(t)$. (1 p)

(c) Determine $X_2(f)$, the Fourier transform of $x_2(t)$:

$$X_2(f) = \int_{-\infty}^{\infty} x_2(t)e^{-j2\pi ft}\, dt.$$

(2 p)

(d) Using the Fourier transform differentiation property

$$\mathcal{F}\left\{\frac{dx(t)}{dt}\right\} = j2\pi f X(f),$$

determine $X_1(f)$ from $X_2(f)$. (2 p)

Problem 3. (a) Let

$$x(t) = e^{-t}u(t), \quad h(t) = u(t),$$

where $u(t)$ is the unit step function. Compute the convolution

$$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)\, d\lambda$$

and sketch $y(t)$ versus t. (3 p)

(b) Let

$$X(f) = \text{rect}\left(\frac{f}{B}\right), \quad B = 200 \text{ kHz},$$

and

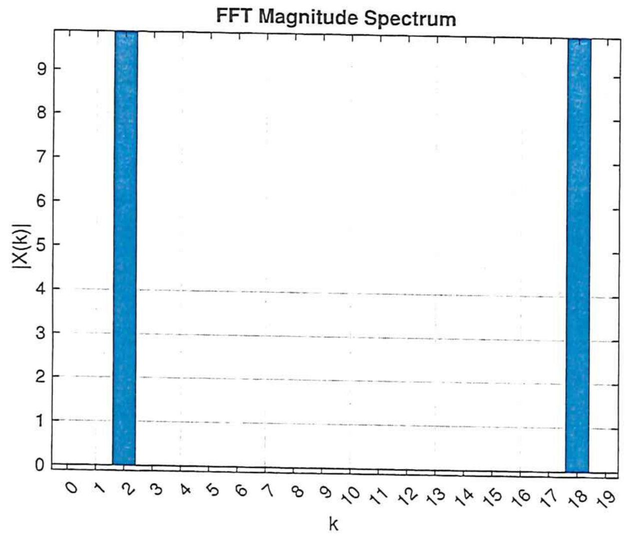
$$H(f) = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0), \quad f_0 = 100 \text{ MHz}.$$

Compute the convolution in the frequency domain:

$$Y(f) = X(f) * H(f)$$

and sketch $|Y(f)|$ versus f. (3 p)

Problem 4. A periodic signal $x(t)$ is sampled using sample interval $T_s = 10$ ms to obtain a sequence $x(0), x(T_s), x(2T_s), \cdots x((N-1)Ts)\}$. An $N = 20$ -point FFT is computed from the samples, and the magnitude spectrum $|X[k]|$ is shown below (index $k = 0, 1, \ldots, 19$).



(a) What is the sampling frequency f_s? (1 p)

(b) What is the Nyquist frequency f_N? (1 p)

(c) Determine the frequencies (in Hz) corresponding to each FFT index k. Express your answer as a table for $k = 0, 1, \ldots, 19$. (2 p)

(d) Determine $\sum_{n=0}^{N-1} |x(nT_s)|^2$ using Parseval's theorem. (2 p)

Problem 5. A single-tone signal

$$x(t) = \cos(2\pi f_0 t)$$

is applied to a nonlinear audio amplifier modeled as

$$y(t) = a\,x(t) + b\,x^3(t),$$

where a and b are constants. Assume $f_0 = 50$ kHz, $a = 2.0$, and $b = -0.5$.

(a) Express $y(t)$ as a cosine Fourier series containing only the fundamental and third harmonic terms.

Hint:

$$\cos^3\theta = \frac{3\cos\theta + \cos 3\theta}{4}.$$

(2 p)

(b) Compute the *third-order harmonic distortion* (THD_3) defined as

$$\mathrm{THD}_3 = \frac{\text{Amplitude of 3rd harmonic}}{\text{Amplitude of fundamental}} \times 100\%.$$

Give your answer in percent. (2 p)

(c) The output is to be passed through a low-pass Butterworth filter to reduce power of the third harmonic $3f_0$ by at least 20 dB. Assume $f_c = 1.5f_0$. The filter amplitude response is

$$A(f) \triangleq |H(f)| = \frac{1}{\sqrt{1 + \left(\frac{f}{f_c}\right)^{2n}}},$$

where f_c is the cutoff frequency and n is the filter order.

Determine minimum integer n to meet the attenuation requirement.

Theorems of the fourier transform	Function	Transform
Linearity	$ax(t)+by(t)$	$aX(f)+bY(f)$
Time delay or time shift	$x(t-a)$	$X(f)e^{-j2\pi fa}$
Scale change	$x(at)$	$\frac{1}{\lvert a\rvert}X(\frac{f}{a})$
Conjugation	$x^*(t)$	$X^*(-f)$
Duality	$X(t)$	$x(-f)$
Frequency shift	$x(t)e^{j2\pi at}$	$X(f-a)$
Linear modulation	$x(t)\cos(2\pi at+b)$	$\frac{e^{jb}X(f-a)+e^{-jb}X(f+a)}{2}$
Differentiation	$\frac{d^n x(t)}{dt^n}$	$(j2\pi f)^n X(f)$
Integration	$\int_{-\infty}^{t} x(u)du$	$\frac{X(f)}{j2\pi f}$
Convolution	$x(t)\otimes y(t)$	$X(f)Y(f)$
Multiplication	$x(t)y(t)$	$X(f)\otimes Y(f)$
Multiplication by t^n	$t^n x(t)$	$(-j2\pi)^{-n}\frac{d^n X(f)}{df^n}$

Fourier transforms	Function	Transform
Rectangular pulse	$\mathrm{rect}(t/a)$	$a\cdot\mathrm{sinc}(af)$
Triangular pulse	$\mathrm{tria}(t/a)$	$a\cdot\mathrm{sinc}^2(af)$
Gaussian pulse	$e^{-\pi(\frac{t}{a})^2}$	$a\cdot e^{-\pi(af)^2}$
One sided exponential pulse	$e^{-t/a}u(t)$	$\frac{a}{1+j2\pi fa}$
Two sided exponential pulse	$e^{-\lvert t\rvert/a}$	$\frac{2a}{1+(2\pi fa)^2}$
Sinc pulse	$\mathrm{sinc}(at)$	$\frac{1}{a}\mathrm{rect}(f/a)$
Constant	a	$a\cdot\delta(f)$
Phasor	$e^{j(2\pi at+b)}$	$e^{jb}\delta(f-a)$
Cosine wave	$\cos(2\pi at+b)$	$\frac{e^{jb}\delta(f-a)+e^{-jb}\delta(f+a)}{2}$
Delayed impulse	$\delta(t-a)$	$e^{-j2\pi fa}$
Step	$u(t)$	$\frac{\delta(f)}{2}+\frac{1}{j2\pi f}$

$$T_0 = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$$

$$e^{j\phi} = \cos(\phi) + j\sin(\phi)$$

$$\sin(\phi) = \frac{1}{2j}(e^{j\phi} - e^{-j\phi})$$

$$\cos(\phi) = \frac{1}{2}(e^{j\phi} + e^{-j\phi})$$

$$\sin^2\phi + \cos^2\phi = 1$$

$$\cos(\phi) = \sin(\phi + \pi/2)$$

$$\sin(\phi) = \cos(\phi - \pi/2)$$

$$\sin(\alpha)\cos(\beta) = \frac{\sin(\alpha-\beta) + \sin(\alpha+\beta)}{2}$$

$$\sin(\alpha)\sin(\beta) = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}$$

$$\cos(\alpha)\cos(\beta) = \frac{\cos(\alpha-\beta) + \cos(\alpha+\beta)}{2}$$

$$x(t)\otimes y(t) = \int_{-\infty}^{\infty} x(\lambda)y(t-\lambda)\,d\lambda = y(t)\otimes x(t)$$

$$x(t) = \sum_{k=-\infty}^{\infty} x_k e^{j2\pi k f_0 t} = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty}\left[\alpha_k\cos(2\pi k f_0 t) + \beta_k\sin(2\pi k f_0 t)\right]$$

$$x_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-j2\pi k f_0 t}dt$$

$$\alpha_k = 2\cdot\mathrm{Re}\{x_k\}, \quad \text{when } x(t)\in\mathbb{R}$$

$$\beta_k = -2\cdot\mathrm{Im}\{x_k\}, \quad \text{when } x(t)\in\mathbb{R}$$

$$X(f) = \mathcal{F}\left\{x(t)\right\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

$$x(t) = \mathcal{F}^{-1}\left\{X(f)\right\} = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

$$X_k = \sum_{n=0}^{N-1} x_n e^{-j2\pi kn/N}$$

$$x_k = \frac{1}{N}\sum_{n=0}^{N-1} X_n e^{j2\pi kn/N}$$

$$f_0 = \frac{1}{N\cdot T_s} = \frac{f_s}{N}$$

$$s = \sigma + j\omega = \sigma + j2\pi f$$

$$X(s) = \mathcal{L}\left\{x(t)\right\} = \int_0^{\infty} x(t)\cdot e^{-st}dt$$

$$d_n = \frac{u_n}{u_1}$$

$$d_{\mathrm{tot}} = \sqrt{\sum_{n=2}^{\infty} d_n^2}$$