Instructions:

- Allowed in the exam: a calculator and the sheet of A5-paper (double-sided) with your own "idea notes".
- Problems are NOT arranged in the order of increasing difficulty. The difficulty level is expressed by the number of points for each problem. You can solve the problems in any order.
- The maximum from the following three problems is 40 points.
- Don't forget to put your name in all the sheets of paper before returning them to the examiner.
- Please return this sheet at the end of the exam.

1. (14 points) A right–hand circularly polarized (RHCP) uniform plane wave traveling in air is incident normally on a flat and smooth water surface at $x = 0$. The water has relative permittivity $\varepsilon_{r2} = 81$ and conductivity $\sigma_2 = 0.1$ S/m. The air region ($x < 0$) is lossless with $\varepsilon_{r1} = 1$, $\sigma_1 = 0$. The frequency is $f = 1$ GHz and the incident electric field is

$$\mathbf{E}^i(x) = \left(\mathbf{a}_y + \mathbf{a}_z e^{j\psi}\right) E_0 e^{-j\beta_0 x},$$

where β_0 is the phase constant in free space and E_0 is a real constant.

(a) Determine the value of ψ so that the wave is RHCP.

(b) Write expressions for the reflected electric and magnetic fields, the reflected electric and magnetic fields, and the transmitted electric and magnetic fields (Hint: Euler form might be useful)

(c) Determine the polarization of the reflected wave.

(d) Determine the percentage of the incident time–average power density that is reflected and transmitted at the interface.

(e) Now suppose the incident RHCP plane wave arrives at the interface at an oblique angle θ_i (measured from the normal). Assume the incident wave approaches the interface from the region $x < 0$ and lies in the x–z plane, with the projection of the incident wavevector onto the interface directed along $+\mathbf{a}_z$.

   i. Determine the directions of the reflected and transmitted wavevectors.

   ii. Determine whether the reflected wave remains circularly polarized. Justify your answer.

   iii. State whether the transmitted wave remains circularly polarized in the lossy water. Justify your answer.

Solution:

(a) Phase ψ for RHCP

At a fixed x, the real fields are

$$E_y(t) = E_0 \cos\omega t, \qquad E_z(t) = E_0 \cos(\omega t + \psi).$$

For circular polarization we need equal magnitudes and a $\pm 90^\circ$ phase difference. For a wave traveling in $+x$, the standard IEEE convention for a RHCP wave is

$$\mathbf{E}^i = \left(\mathbf{a}_y + j\mathbf{a}_z\right) E_0 e^{-j\beta_0 x},$$

so that E_z leads E_y by 90°. Hence

$$\boxed{\psi = \frac{\pi}{2}}.$$

Thus we will use

$$\mathbf{E}^i(x) = \left(\mathbf{a}_y + j\mathbf{a}_z\right) E_0 e^{-j\beta_0 x}.$$

Notice we also accept $\psi = -\frac{\pi}{2}$ which follows the convention in the lecture notes; all the expressions written consistently with either of the angles are counted correctly.

(b) Reflected fields

Material parameters for water:

The complex permittivity of water is

$$\varepsilon_c = \varepsilon_0\varepsilon_{r2} - j\frac{\sigma_2}{\omega},$$

and its intrinsic impedance and propagation constant are

$$\eta_2 = \sqrt{\frac{j\omega\mu_0}{\sigma_2 + j\omega\varepsilon_0\varepsilon_{r2}}}, \qquad \gamma_2 = \sqrt{j\omega\mu_0\big(\sigma_2 + j\omega\varepsilon_0\varepsilon_{r2}\big)} = \alpha_2 + j\beta_2.$$

Numerically at $f = 1$ GHz,

$$\eta_0 \approx 377\ \Omega, \qquad \eta_2 \approx 41.85 + j0.46\ \Omega.$$

The normal–incidence reflection coefficient (in air) is

$$\Gamma = \frac{\eta_2 - \eta_0}{\eta_2 + \eta_0} \approx -0.800 + j\,0.002,$$

and the transmission coefficient is

$$T = 1 + \Gamma = \frac{2\eta_2}{\eta_2 + \eta_0} \approx 0.200 + j\,0.002.$$

The reflected wave travels in the $-\mathbf{a}_x$ direction with the same transverse polarization vector; only its complex amplitude changes by Γ. Thus

$$\boxed{\mathbf{E}^r(x) = \Gamma\big(\mathbf{a}_y + j\mathbf{a}_z\big)E_0 e^{+j\beta_0 x}, \quad x < 0.}$$

For the reflected magnetic field we use

$$\mathbf{H}^r = \frac{1}{\eta_0}\hat{\mathbf{k}}_r \times \mathbf{E}^r, \quad \hat{\mathbf{k}}_r = -\mathbf{a}_x,$$

so

$$\mathbf{H}^r(x) = \frac{\Gamma E_0}{\eta_0}(-\mathbf{a}_x) \times \big(\mathbf{a}_y + j\mathbf{a}_z\big)e^{+j\beta_0 x} = \frac{\Gamma E_0}{\eta_0}\big(j\mathbf{a}_y - \mathbf{a}_z\big)e^{+j\beta_0 x}.$$

Hence

$$\boxed{\mathbf{H}^r(x) = \frac{\Gamma E_0}{\eta_0}\big(j\mathbf{a}_y - \mathbf{a}_z\big)e^{+j\beta_0 x}, \quad x < 0.}$$

Transmitted fields:

The transmitted wave propagates in medium 2 with intrinsic impedance η_2 and complex propagation constant $\gamma_2 = \alpha_2 + j\beta_2$. The transmission coefficient at normal incidence is

$$T = 1 + \Gamma = \frac{2\eta_2}{\eta_2 + \eta_0}.$$

Thus the transmitted electric field is

$$\boxed{\mathbf{E}^t(x) = T\big(\mathbf{a}_y + j\mathbf{a}_z\big)E_0\, e^{-\gamma_2 x}, \qquad x > 0.}$$

The transmitted magnetic field is

$$\boxed{\mathbf{H}^t(x) = \frac{TE_0}{\eta_2}\big(\mathbf{a}_z - j\mathbf{a}_y\big)e^{-\gamma_2 x}, \qquad x > 0.}$$

(c) Polarization of the reflected wave

The reflected electric field is proportional to $\mathbf{a}_y + j\mathbf{a}_z$ but propagates in the $-\mathbf{a}_x$ direction. The ratio of components is unchanged:

$$\frac{E_z^r}{E_y^r} = j,$$

so the tip of $\mathbf{E}^r$ still traces a circle with equal magnitudes and a $+90°$ phase lead of E_z over E_y.

However, reversing the direction of propagation reverses the sense of rotation. A RHCP wave traveling in $+\mathbf{a}_x$ becomes *left–hand circularly polarized* (LHCP) when it propagates in $-\mathbf{a}_x$ with the same transverse vector.

$$\boxed{\text{The reflected wave is LHCP.}}$$

(d Reflected and transmitted power densities

For a uniform plane wave in a lossless medium,

$$\langle S\rangle = \frac{|E|^2}{2\eta_1}.$$

Thus the incident and reflected time–average power densities in air are

$$\langle S_i\rangle = \frac{|E_0|^2}{2\eta_0}, \qquad \langle S_r\rangle = \frac{|\Gamma|^2|E_0|^2}{2\eta_0}.$$

Therefore

$$\frac{\langle S_r\rangle}{\langle S_i\rangle} = |\Gamma|^2 \approx (0.800)^2 \approx 0.64.$$

So about 64% of the incident power is reflected.

The transmitted power density at the interface $x = 0^+$ is

$$\langle S_t(0^+)\rangle = \langle S_i\rangle - \langle S_r\rangle = (1-|\Gamma|^2)\langle S_i\rangle,$$

so

$$\frac{\langle S_t(0^+)\rangle}{\langle S_i\rangle} = 1-|\Gamma|^2 \approx 1-0.64 = 0.36.$$

(e) Oblique incidence

(i) Directions of the reflected and transmitted wavevectors

The incident wavevector in the x–z plane is

$$\mathbf{k}_i = k_0\big(\sin\theta_i\,\mathbf{a}_z - \cos\theta_i\,\mathbf{a}_x\big).$$

$$\mathbf{k}_r = k_0\big(\sin\theta_i\,\mathbf{a}_z + \cos\theta_i\,\mathbf{a}_x\big).$$

$$\mathbf{k}_t = k_2\big(\sin\theta_t\,\mathbf{a}_z - \cos\theta_t\,\mathbf{a}_x\big).$$

(ii) Circular polarization of the reflected wave

No. A circularly polarized wave can be written as a superposition of TE and TM components with equal amplitudes and a $\pm 90°$ phase difference:

$$\mathbf{E}^{\text{RHCP}} = \frac{1}{\sqrt{2}}\big(\mathbf{E}_{\text{TE}} + j\,\mathbf{E}_{\text{TM}}\big).$$

At oblique incidence, the reflection coefficients for TE and TM waves differ:

$$\Gamma_{\text{TE}} \neq \Gamma_{\text{TM}}.$$

So the reflected field becomes

$$\mathbf{E}^r \propto \Gamma_{\text{TE}}\mathbf{E}_{\text{TE}} + j\,\Gamma_{\text{TM}}\mathbf{E}_{\text{TM}},$$

which no longer has equal magnitudes or a 90° phase difference between the two components.

Therefore the reflected wave becomes *elliptically polarized*.

(iii) Circular polarization of the transmitted wave

No, in general it does not.

For the transmitted wave, the TE and TM transmission coefficients also differ:

$$T_{\mathrm{TE}} \neq T_{\mathrm{TM}}.$$

Thus the relative amplitudes and phases of the transmitted TE/TM components are unequal, producing an elliptically polarized wave:

$$\mathbf{E}^t \propto T_{\mathrm{TE}}\mathbf{E}_{\mathrm{TE}} + j\,T_{\mathrm{TM}}\mathbf{E}_{\mathrm{TM}}.$$

In addition, the water is lossy, so the attenuation constants for TE and TM components differ slightly, further distorting the polarization as the wave propagates.

2. (12 points) We have a lossless coaxial line made of a inner conductive wire with radius $a = 0.36$ [mm], surrounded by a dielectric layer with $\varepsilon_r = 2.25$ ($n = 1.5$ with $\mu_r = 1$), and a total outer radius $b = 2.35$ [mm] (without shielding). One end of the cable is connected to a 50 [Ω] load.

   (a) Calculate the characteristic impedance Z_0 and the propagation factor γ, using a reference frequency of 1 GHz. **Hint:** $\ln 6.5 \approx 1.87$.
   (b) Determine the Standing Wave Ratio (SWR) produced by the load.
   (c) Estimate the percentage of power reflected and absorbed by the load.
   (d) Calculate the input impedance and the reflection coefficient at the other end of the line if the coaxial cable has a length of 2.5 [m].

Solution:

(a)

In order to calculate the characteristic impedance and the propagation factor, first we need to determine the transmission line components of the equivalent model. By assuming that the coaxial cable is lossless, we have that $R = 0$ and $G = 0$. Therefore, the only components we need to find is the transmission line inductance L and capacitance C per unit of length. For that, we can use the expressions

$$L = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)$$

$$C = \frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)}$$

We can find the value of the characteristic impedance using the expression

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{L}{C}}$$

$$Z_0 = \sqrt{\frac{\frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)}{\frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)}}}$$

$$Z_0 = \frac{1}{2\pi}\ln\left(\frac{b}{a}\right)\sqrt{\frac{\mu}{\varepsilon}}$$

Using the provided values, we can find that $\boxed{Z_0 \approx 75\ [\Omega]}$.

For the propagation factor $\gamma = \alpha + j\beta$, we can use the expression

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{(j\omega)^2 LC}$$

Therefore, we can conclude that, since the coaxial cable is lossless, their propagation parameters are

$$\alpha = 0$$

$$\beta = \omega\sqrt{LC}$$

$$\beta = \omega\sqrt{\frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)\frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)}}$$

$$\beta = \omega\sqrt{\mu\varepsilon}$$

$$\beta = \frac{\omega\sqrt{\varepsilon_r}}{c}$$

The propagation constant @ 1 GHz is equal to $\boxed{\beta = 31.42 \text{ [rad/m]}}$.

(b)

In order to calculate the SWR, first we need to find the reflection coefficient produced by the $Z_L = 50$ [Ω] load. This value can be found out using the expression

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\Gamma_L = \frac{50 - 75}{50 + 75}$$

$$\boxed{\Gamma_L = -\frac{1}{5}}$$

Then, the SWR is calculated using the formula

$$\text{SWR} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$$

$$\text{SWR} = \frac{1 + 0.2}{1 - 0.2}$$

$$\boxed{\text{SWR} = \frac{3}{2}}$$

(c)

The reflected power is proportional to $|\Gamma_L|^2 = 0.04$. Therefore, the percentage of reflected power is approximately 4%. The rest of the power (96%) is absorbed by the load.

(d)

The input impedance for a load connected to a transmission line with length l reads

$$Z_{in} = Z_0\frac{Z_L + jZ_0\tan(\beta l)}{Z_0 + jZ_L\tan(\beta l)}$$

For a 2.5 [m] long cable, we can find that $\tan(\beta l) \approx 0.01$, then

$$Z_{in} = 75\frac{50 + 75j \cdot 0.01}{75 + 50j \cdot 0.01}$$

$$\boxed{Z_{in} = 50 + 0.42j \text{ [Ω]}}$$

And the input reflection coefficient read

$$\Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} - Z_0}$$

$$\boxed{\Gamma_{in} \approx -0.190 + 0.004j}$$

3. (14 points) Consider an air-filled rectangular waveguide of dimensions a and b, with $a > b > 1$. The waveguide is terminated at $z = d$ by a perfect conductor, forming a rectangular cavity of length d. A coaxial probe is inserted through the wall at a point where the cavity supports a strong electric field magnitude. The coaxial structure has inner conductor radius a_c and outer conductor radius b_c, and is also air-filled.

(a) Which of the following modes propagates or is evanescent at an operating frequency f_0, assuming we have $f_0 > f_{c,\mathrm{TE}_{10}}$, $a = 2b$ and $f_0 = \frac{c_0}{2a}$.

- TE_{10}
- TM_{11}
- TE_{20}

Give a brief conceptual explanation (in 1-2 sentences) for each classification. What physical quantity determines when a mode becomes evanescent?

(b) For a mode operated at slightly above cutoff $f_0 = f_c(1+\Delta)$, with $0 < \Delta \ll 1$, derive an expression for the guide wavelength λ_g. How does λ_g behave physically near the cutoff? Briefly explain why.

(c) When the guide is capped at $z = 0$ and $z = d$, the system becomes a resonant cavity. Suppose the cavity must support a TE_{101} resonance at a design frequency f_0. Starting from the separated-variable fields for waveguide TE_{mn} modes, solve for the required cavity length d in terms of f_0, a, and b.

(d) An air-filled coaxial transmission line with inner radius a_c, outer radius b_c, carries a TEM wave. Derive the electric and magnetic fields for the TEM mode.

In one paragraph, explain how and why a coaxial probe couples to a cavity TE mode, including the importance of probe orientation relative to the local electric field.

Solution:

(a)

The cutoff frequency is

$$f_{c,mn} = \frac{c_0}{2\pi} k_{c,mn} = \frac{c_0}{2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = \frac{c_0}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

A mode propagates (for real β) when $f_0 > f_c$. If $f_0 < f_c$ the longitudinal wavenumber becomes imaginary and the mode is evanescent (exponentially decays along z). Thus

- TE_{10} - Propagating. By hypothesis $f_0 > f_{c,\mathrm{TE}_{10}}$, so $\beta = \sqrt{k_0^2 - k_{c,\mathrm{TE}_{10}}^2}$ is real and the mode propagates.
- TM_{11} - Evanescent. We have $f_0 = \frac{c_0}{2a} < f_{c,\mathrm{TM}_{11}} = \frac{c_0}{2}\sqrt{(1/a)^2 + (1/b)^2} = \frac{\sqrt{5}c_0}{2a}$.
- TE_{20} - Evanescent. We also have TE_{20} has cutoff $f_{c,\mathrm{TE}_{20}} = \frac{c_0}{2}\sqrt{(2/a)^2 + (0/b)^2} = \frac{c_0}{a} > f_0 = \frac{c_0}{2a}$.

A mode becomes evanescent when the operating free-space wavenumber k_0 is smaller than the transverse cutoff wavenumber k_c. Physically this means the wave cannot sustain a real longitudinal propagation constant β (it becomes imaginary), so the mode's fields decay exponentially along the guide instead of carrying power.

(b)

Consider near cutoff, let $f_0 = f_c(1+\Delta)$. Then $k_0 = (2\pi f_0)/c_0 = k_c(1+\Delta)$ because $k \propto f$, so

$$\beta = \sqrt{k_0^2 - k_c^2} = k_c\sqrt{(1+\Delta)^2 - 1} = k_c\sqrt{2\Delta + \Delta^2}.$$

For $\Delta \ll 1$ we can neglect Δ^2, giving $\beta \approx k_c\sqrt{2\Delta}$.

The guide wavelength is

$$\lambda_g = \frac{2\pi}{\beta} \approx \frac{2\pi}{k_c\sqrt{2\Delta}}$$

But the free-space wavelength at f_0 is $\lambda_0 = 2\pi/k_0 \approx 2\pi/(k_c(1+\Delta)) \approx 2\pi/k_c$ to first order in small Δ. Therefore,

$$\boxed{\lambda_g \approx \lambda_0 \frac{1}{\sqrt{2\Delta}}}$$

Near cutoff $\beta \to 0$, which means phase varies very slowly along z—the guided wave "stretches" and the guide wavelength $\lambda_g = 2\pi/\beta$ becomes very large. This is because most of the energy is stored in the transverse fields and the longitudinal phase propagation is very slow when the operating frequency is barely above the transverse resonance (cutoff).

(c)

When the guide is closed by perfect-conducting end-plates at $z = 0$ and $z = d$, standing waves form along z with boundary conditions enforcing tangential electric fields to vanish at the conducting end-plates.

For modal separation, the field dependence along z is $\sin(k_z z)$ or $\cos(k_z z)$. Imposing $E_t(z = 0) = 0$ and $E_t(z = d) = 0$ typically leads to

$$k_z = \frac{p\pi}{d}, \quad p = 1, 2, 3, \ldots$$

The full wavenumber for the cavity resonance is

$$k_{r,mnp}^2 = k_{c,mn}^2 + k_z^2,$$

where $k_{c,mn}^2 = (m\pi/a)^2 + (n\pi/b)^2$. Therefore

$$k_{r,mnp} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2}.$$

The resonant frequency f_{mnp} satisfies $k_{r,\text{mnp}} = 2\pi f_{\text{mnp}}/c_0$, so

$$f_{mnp} = \frac{c_0}{2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2}.$$

This is the standard formula for a rectangular cavity.

For TE_{101}, $m = 1, n = 0, p = 1$. Plugging into the resonant-frequency equation, we have

$$2\pi f_0/c_0 = \sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{0}{b}\right)^2 + \left(\frac{\pi}{d}\right)^2} = \sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{d}\right)^2}$$
$$\Leftrightarrow \left(\frac{\pi}{d}\right)^2 = \left(\frac{2\pi f_0}{c_0}\right)^2 - \left(\frac{\pi}{a}\right)^2$$

So,

$$\boxed{d = \frac{\pi}{\sqrt{\left(\frac{2\pi f_0}{c_0}\right)^2 - \left(\frac{\pi}{a}\right)^2}}}.$$

(d)

A TEM mode has purely transverse electric and magnetic fields with no longitudinal components in ideal coax. Start from electrostatics for the potential of a concentric line: the radial electric field $E_r(r)$ in cylindrical coordinates arises from a line charge density λ on the inner conductor and opposite on the outer conductor.

Gauss law (for radius r satisfying $a_c < r < b_c$)

$$\oint \mathbf{E} \cdot d\mathbf{A} = E_r(2\pi r L) = \frac{\lambda L}{\varepsilon_0},$$

So

$$E_r(r) = \frac{\lambda}{2\pi\varepsilon_0 r}.$$

Potential difference between inner and outer conductor

$$V = -\int_{a_c}^{b_c} \mathbf{E} \cdot d\mathbf{r} = \int_{a_c}^{b_c} \frac{\lambda}{2\pi\varepsilon_0 r} dr = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{b_c}{a_c}\right).$$

Hence the line charge is

$$\lambda = \frac{2\pi\varepsilon_0 V}{\ln(b_c/a_c)}.$$

Substitute back to get

$$\boxed{E_r(r) = \frac{\lambda}{2\pi\varepsilon_0 r} = \frac{V}{r\ln(b_c/a_c)}} \quad (a_c < r < b_c).$$

Magnetic field from Ampere's law for current $\boldsymbol{I}$ on the inner conductor (and return on outer)

$$\oint \mathbf{H}\cdot d\boldsymbol{l} = H_\phi(2\pi r) = I \quad \Rightarrow \quad \boxed{H_\phi(r) = \frac{I}{2\pi r}} \quad (a_c < r < b_c).$$

These are the standard TEM field expressions for a lossless air-filled coax.

A coaxial probe (a small inner conductor pin extending into the cavity from the outer wall) couples to the cavity by sampling the local electric field and/or inducing a local current on the cavity walls. A small, electrically-short probe oriented with its axis parallel to the local electric field (i.e., the probe tip in the direction of the E-field vector) acts as an electrical dipole that perturbs the cavity fields and exchanges energy: it extracts power when the probe samples an anti-node of the cavity electric field, and it couples minimally when placed at a node. For TE modes (which have dominant transverse electric fields and significant H-field), an electric probe should be placed where the transverse E-field is strong (and oriented to match that field component). Conversely, a magnetic loop probe (small loop) couples to the transverse magnetic field by sampling the magnetic flux (and is placed where $|\mathbf{H}|$ is large). The coupling strength depends on probe length, penetration depth, and orientation; too deep a probe overcouples and loads the cavity, while a shallow probe provides weak coupling. Thus correct positioning and orientation allow controlled coupling of the coax to the TE cavity mode.