Fourier Theory / Hytönen

Course exam 17.2.2025

Solve any **four (4)** of the following problems. The choice is yours, all are worth the same. If you solve more than four problems, only four will be graded, and not necessarily your best four.

In any problem that you choose, you should write down the definitions of the objects and symbols appearing in the statement of the problem: this can earn you some points, even if you don't solve the full problem. If you use some non-trivial theorem in your argument, you should mention this explicitly. In general, you are allowed to use any results from the course, except of course the ones that you are specifically asked to prove.

Problem 1. Consider the statement: $(\log|x|)' = \mathrm{p.v.}\,\frac{1}{x}$ in the sense of tempered distributions. Explain what it means, and prove it. (Here, we are working in one dimension: $x \in \mathbb{R}$.)

Problem 2. Sketch the solution of the periodic heat equation

$$\begin{cases} \partial_t u(t,x) &= \triangle_x u(t,x), \quad \text{for} \quad t>0,\ x\in\mathbb{T}^d \\ u(0,x) &= f(x), \qquad\quad \text{for} \quad x\in\mathbb{T}^d \end{cases}$$

with the help of Fourier series. Give a formula for the periodic heat kernel and explain why it is positive everywhere.

Problem 3. Let μ be a finite measure on $\mathbb{R}$. Define its Fourier transform $\hat{\mu}$ and prove that

$$\mu((-\frac{1}{\pi r}, \frac{1}{\pi r})^c) \leq \frac{1}{r}\int_{-r}^{r} (\hat{\mu}(0) - \hat{\mu}(\xi))\,\mathrm{d}\xi.$$

for every $r > 0$.

Problem 4. Let $\rho \in L^2(\mathbb{R}^d)$. Prove that the set of translates $\{\rho(\cdot - k) : k \in \mathbb{Z}^d\}$ is orthonormal if and only if

$$\sum_{\ell\in\mathbb{Z}^d} |\hat{\rho}(\xi+\ell)|^2 = 1$$

for almost all $\xi \in \mathbb{R}^d$.

Problem 5. In this course, we have defined the Fourier transform by

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-i2\pi x\cdot\xi}\,\mathrm{d}x.$$

Discuss the advantages and disadvantages of this choice of normalisation.

(Warning! The usual caveats of an "essay" problem! When you are asked to prove something, you know when you are done. Here, the notion of a "complete solution" is not so well defined.)