

MS-E1461 Hilbert spaces (Aalto University, Turunen / Zacchini)
Examination on Wednesday 14.10.2026, at 13:00-16:00

Points also for good effort! No literature, no calculators, etc. If you refer to any results in the course, please name them.

Note: In the following, H is a complex Hilbert space, with the inner product $(u, v) \mapsto \langle u, v \rangle$ and the norm $u \mapsto \|u\| = \langle u, u \rangle^{1/2}$.

1. (a) Describe the vector space operations, the inner product and the norm of the complex Hilbert space $\ell^2 := \ell^2(\mathbb{Z}^+)$.
(b) Show that $u, v \in H$ are orthogonal if and only if

$$\|u\| \leq \|u + \lambda v\| \quad (1)$$

for every $\lambda \in \mathbb{C}$.

2. Let $v, w \in H$ be linearly independent (in other words, $\lambda v + \mu w = 0 \in H$ if and only if $\lambda = \mu = 0 \in \mathbb{C}$). Let

$$M := \{\lambda v + \mu w : \lambda, \mu \in \mathbb{C}\} \quad (2)$$

Use the Gram-Schmidt algorithm to find an orthonormal basis for M .

Use this orthonormal basis to write a formula $P_M u = \dots$ for the orthogonal projection $P_M : H \rightarrow H$ onto the closed vector subspace M .

3. Let $\ell^2 = \ell^2(\mathbb{Z}^+)$. Let us define $A : \ell^2 \rightarrow \ell^2$ by

$$Au(k) := (i^k + 1/k) u(k), \quad (3)$$

where $i \in \mathbb{C}$ is the imaginary unit. Find the norm $\|A\|$. Find the adjoint $A^* \in \mathcal{B}(\ell^2)$. Show that A is normal. Is A compact? Justify your answer!

4. Let $A : \ell^2 \rightarrow \ell^2$ as in (3) above.
Find all the eigenvalues and all the eigenvectors of A .
Find the spectrum $\sigma(A) \subset \mathbb{C}$.
Calculate

$$((\lambda I - A)^{-1}v)(k) = \dots \quad (4)$$

when $v(k) := 1/k$ for all $k \in \mathbb{Z}^+$, and $\lambda \notin \sigma(A)$.