PHYS-E0461 Introduction to plasma physics for fusion and space applications
First examination 12.10.2026

If any of the problems ask you to calculate numerical values, it is enough to give rough estimates. Values of common physical constants, together with basic physics formulas, are given in the appendix.

1. **(Plasma phenomena and general aspects)** (6p)
   Give short (not more than 1/3 page for each) but complete answers to following three questions:
   (a) Explain what is meant by quasineutrality (in relation to plasmas)? (1p)
   (b) Explain what is the plasma parameter and how it relates to the Debye sphere. (1p)
   (c) During the lectures several plasma waves were introduced. Why there are so many waves compared to neutral gas (that only carries the sound wave)? Give an example of a plasma wave. (1p)
   (d) Let's make sure you understand that the existence of plasma requires that the plasma parameter Λ has to satisfy the condition $\Lambda \gg 1$.
   The basic requirement is that the electrons and ions should stay apart, i.e., not recombine. This means that their kinetic energy E_{kin} should greatly exceed their interaction energy, here assumed to be purely electrostatic, $E_{pot} \sim q_1 q_2/d$, where d stands for the distance between charges q_1 and q_2. Note also that, since we are doing order-of-magnitude estimates, we neglect algebraic junk, such as factors of 4π. Consider now only electrons with density n_e and temperature T_e, so that $q_1 = q_2 = e$. Show that this physical requirement equals the condition $\Lambda \sim n_e \cdot \lambda_D^3 \gg 1$. (3p)

2. **(Charged particle motion in magnetic and electric fields.)** (6p)
   Integrate the charged particle motion (find the position as a function of time) in a homogeneous magnetic field, $\boldsymbol{B} = B_0\hat{\boldsymbol{z}}$,
   (a) for $\boldsymbol{E} = \mathbf{0}$ (no electric field). (3p)
   (b) for $\boldsymbol{E} = E_0\hat{\boldsymbol{x}}$. (3p)

3. **(About the cyclotron waves.)** (6p)

   In the rotating basis, the wave equation of parallel waves ($\mathbf{B} = B_0\mathbf{e}_z$, $\mathbf{k} = k\mathbf{e}_z$) diagonalises:

$$\begin{aligned}(L - n^2)\tilde{E}_+ &= 0,\\ (R - n^2)\tilde{E}_- &= 0,\\ P\tilde{E}_z &= 0,\end{aligned}$$

   with $\tilde{E}_\pm = (\tilde{E}_x \pm i\tilde{E}_y)/\sqrt{2}$.
   (a) For the pure L-wave to exist, the R-wave must be absent ($\tilde{E}_- = 0$), and vice-versa. Assuming the wave is monochromatic (a single pair of $\mathbf{k}$, ω values), use this condition to show that the L/R-waves are left/right-circular polarised. (3p)
   (b) Discuss qualitatively the implications of this result on the waves' interaction with gyrating charged particles (resonances). (3p)

4. **(From distribution function to real life)** (6p)
   Calculate the first three velocity moments of the distribution function $f_s(\mathbf{x}, \mathbf{v}; t)$, i.e., the 0th, 1st and 2nd moments. Here, the 2nd moment refers to v^2, not the matrix $\mathbf{v} \otimes \mathbf{v}$. Use the 2nd moment to show that the energy density is given by $\frac{3}{2}nT + \frac{1}{2}mnV^2$, where V is the magnitude of the bulk velocity $\mathbf{V}$, $\frac{3}{2}nT$ is the thermal energy density, and $\frac{1}{2}mnV^2$ is the kinetic energy density. The velocity space dependence is given by the *shifted* Maxwellian distribution $f_M(\mathbf{v}) = \left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{m|\mathbf{v}-\mathbf{V}(\mathbf{r})|^2}{2T}\right)$ (normalized to unity).

Appendix

1 Helpful physics formulas

Debye length	$\lambda_D^2 = \dfrac{\epsilon_0 T_e}{e^2 n_e}$	(1)
Plasma parameter	$\Lambda = \dfrac{4}{3} n \pi \lambda_D^3$	(2)
Plasma frequency	$\omega_p^2 = \dfrac{e^2 n}{m \epsilon_0}$	(3)
Larmor frequency	$\Omega = \dfrac{qB}{m}$	(4)
Larmor radius	$r_L = \dfrac{m v_\perp}{qB}$	(5)
Magnetic moment	$\mu = \dfrac{m v_\perp^2}{2B}$	(6)
$\boldsymbol{E} \times \boldsymbol{B}$ drift	$\boldsymbol{v}_{\boldsymbol{E}\times\boldsymbol{B}} = \dfrac{\boldsymbol{E} \times \boldsymbol{B}}{B^2}$	(7)
Gradient drift	$\boldsymbol{v}_{\nabla B} = \pm \dfrac{1}{2} v_\perp r_L \dfrac{\boldsymbol{B} \times \nabla B}{B^2}$	(8)
Diamagnetic drift	$\boldsymbol{v}_D = -\dfrac{\nabla p \times \boldsymbol{B}}{qnB^2}$	(9)
Maxwell-Boltzmann	$f(\boldsymbol{v}) = \left(\dfrac{m}{2\pi T}\right)^{3/2} \exp\left(-\dfrac{mv^2}{2T}\right)$	(10)
	$h(E) = \dfrac{2}{T^{3/2}} \sqrt{\dfrac{E}{\pi}} e^{-E/T}$	(11)
Convective derivative	$\dfrac{\mathrm{d}n(\boldsymbol{r},t)}{\mathrm{d}t} = \dfrac{\partial n(\boldsymbol{r},t)}{\partial t} + \boldsymbol{v} \cdot \nabla n(\boldsymbol{r},t)$	(12)
Collision frequency	$\nu = \dfrac{e^4 \ln \Lambda}{4\pi \epsilon_0^2 \sqrt{m}} \dfrac{n}{T^{3/2}}$	(13)
Gauss's law	$\nabla \cdot \boldsymbol{E} = \dfrac{\rho}{\epsilon_0}$	(14)
Gauss's law for magnetism	$\nabla \cdot \boldsymbol{B} = 0$	(15)
Maxwell-Faraday equation	$\nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$	(16)
Ampère's circuital law	$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \dfrac{1}{c^2} \dfrac{\partial \boldsymbol{E}}{\partial t}$	(17)
Sound speed in ideal gases	$v_s = \sqrt{\dfrac{E_{\text{therm}}}{m}}$	(18)
Sound speed in plasma	$v_s = \sqrt{\dfrac{E_{\text{therm, e}}}{M} + \dfrac{\gamma_i E_{\text{therm, i}}}{M}}$	(19)

2 Constants

- vacuum permittivity $\epsilon_0 = 8.9 \cdot 10^{-12}$ C/Vm
- magnetic permeability $\mu_0 = 4\pi \times 10^{-7}$ Tm/A
- speed of light $c = 3 \times 10^8$ m/s
- elementary charge $e = 1.6 \cdot 10^{-19}$ C
- electron mass $m_e = 9.1 \cdot 10^{-31}$ kg
- proton mass $m_p = 1.7 \times 10^{-27}$ kg

3 Vector identities

1. $\boldsymbol{A} \cdot \boldsymbol{B} \times \boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B} \cdot \boldsymbol{C} = \boldsymbol{B} \cdot \boldsymbol{C} \times \boldsymbol{A}$
2. $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{C} \times \boldsymbol{B}) \times \boldsymbol{A} = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B})$
3. $(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) = (\boldsymbol{A} \cdot \boldsymbol{C})(\boldsymbol{B} \cdot \boldsymbol{D}) - (\boldsymbol{A} \cdot \boldsymbol{D})(\boldsymbol{B} \cdot \boldsymbol{C})$
4. $(\boldsymbol{A} \times \boldsymbol{B}) \times (\boldsymbol{C} \times \boldsymbol{D}) = \boldsymbol{C}(\boldsymbol{A} \times \boldsymbol{B} \cdot \boldsymbol{D}) - \boldsymbol{D}(\boldsymbol{A} \times \boldsymbol{B} \cdot \boldsymbol{C})$

5. $\nabla \cdot (f\boldsymbol{A}) = f(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot (\nabla f)$
6. $\nabla \times (f\boldsymbol{A}) = f(\nabla \times \boldsymbol{A}) - \boldsymbol{A} \times (\nabla f)$
7. $\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})$
8. $\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A})$
9. $\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A}) + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A}$
10. $\nabla \cdot (\boldsymbol{A}\boldsymbol{B}) = (\nabla \cdot \boldsymbol{A})\boldsymbol{B} + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}$
11. $\nabla \cdot (f\mathbf{T}) = \nabla f \cdot \mathbf{T} + f\nabla \cdot \mathbf{T}$
12. $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$
13. $\nabla^2 \boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla \times \nabla \times \boldsymbol{A}$
14. $\nabla(fg) = f\nabla g + g\nabla f$
15. $\nabla \cdot (\nabla f \times \nabla g) = 0$
16. $\nabla \cdot \nabla f = \nabla^2 f$
17. $\nabla \times \nabla f = 0$

18. $\int\limits_V \nabla f \, \mathrm{d}V = \int\limits_S f \, \mathrm{d}\boldsymbol{S}$
19. $\int\limits_V \nabla \times \boldsymbol{A} \, \mathrm{d}V = \oint\limits_S \mathrm{d}\boldsymbol{S} \times \boldsymbol{A}$
20. $\int\limits_S \mathrm{d}\boldsymbol{S} \cdot \nabla \times \boldsymbol{A} = \oint\limits_C \mathrm{d}\boldsymbol{l} \cdot \boldsymbol{A}$
21. $\oint\limits_C \mathrm{d}\boldsymbol{l} \times \boldsymbol{A} = \int\limits_S (\mathrm{d}\boldsymbol{S} \times \nabla) \times \boldsymbol{A}$

4 Curvilinear coordinates

Cylindrical Coordinates (r, φ, z)

1. $\nabla f = \dfrac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \dfrac{1}{r}\dfrac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}} + \dfrac{\partial f}{\partial z}\hat{\boldsymbol{z}}$
2. $\nabla \cdot \boldsymbol{A} = \frac{1}{r}\dfrac{\partial}{\partial r}(rA_r) + \dfrac{1}{r}\dfrac{\partial A_\varphi}{\partial \varphi} + \dfrac{\partial A_z}{\partial z}$
3. $\nabla \times \boldsymbol{A} = \left(\frac{1}{r}\dfrac{\partial A_z}{\partial \varphi} - \dfrac{\partial A_\varphi}{\partial z}\right)\hat{\boldsymbol{r}} + \left(\dfrac{\partial A_r}{\partial z} - \dfrac{\partial A_z}{\partial r}\right)\hat{\boldsymbol{\varphi}} + \frac{1}{r}\left(\dfrac{\partial}{\partial r}(rA_\varphi) - \dfrac{\partial A_r}{\partial \varphi}\right)\hat{\boldsymbol{z}}$
4. $\nabla^2 f = \frac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial f}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 f}{\partial \varphi^2} + \dfrac{\partial^2 f}{\partial z^2}$

Spherical Coordinates (r, θ, φ)

5. $\nabla f = \dfrac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \dfrac{1}{r}\dfrac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \dfrac{1}{r\sin\theta}\dfrac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}}$
6. $\nabla \cdot \boldsymbol{A} = \frac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 A_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \theta}(\sin\theta A_\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial A_\varphi}{\partial \varphi}$
7. $\nabla \times \boldsymbol{A} = \frac{1}{r\sin\theta}\left(\dfrac{\partial}{\partial \theta}(\sin\theta A_\varphi) - \dfrac{\partial A_\theta}{\partial \varphi}\right)\hat{\boldsymbol{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\dfrac{\partial A_r}{\partial \varphi} - \dfrac{\partial}{\partial r}(rA_\varphi)\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\dfrac{\partial}{\partial r}(rA_\theta) - \dfrac{\partial A_r}{\partial \theta}\right)\hat{\boldsymbol{\varphi}}$
8. $\nabla^2 f = \frac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial f}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta\dfrac{\partial f}{\partial \theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 f}{\partial \varphi^2}$

5 Gaussian Integrals

Definite integral relations of Gaussian integrals

1. $\int_0^\infty e^{-ax^2}\mathrm{d}x = \frac{1}{2}\left(\frac{\pi}{a}\right)^{1/2}$
2. $\int_{-\infty}^\infty e^{-ax^2}\mathrm{d}x = \left(\frac{\pi}{a}\right)^{1/2}$
3. $\int_{-\infty}^\infty e^{-ax^2}e^{-2bx}\mathrm{d}x = \left(\frac{\pi}{a}\right)^{1/2}e^{\frac{b^2}{a}}$ for $a > 0$
4. $\int_{-\infty}^\infty xe^{-a(x-b)x^2}\mathrm{d}x = b\left(\frac{\pi}{a}\right)^{1/2}$
5. $\int_{-\infty}^\infty x^2e^{-ax^2}\mathrm{d}x = \frac{1}{2}\left(\frac{\pi}{a^3}\right)^{1/2}$
6. $\int_0^\infty x^n e^{-ax^2}\mathrm{d}x = \begin{cases} \dfrac{1}{2}\Gamma\left(\dfrac{n+1}{2}\right)\Big/a^{(n+1)/2} & a > 0 \\ \dfrac{(2k-1)!!}{2^{k+1}a^k}\sqrt{\dfrac{\pi}{a}} & n = 2k, a > 0 \\ \dfrac{k!}{2a^{k+1}} & n = 2k+1, a > 0 \end{cases}$

6 Useful Fourier transforms

1. $\mathcal{F}\left[\dfrac{\partial f}{\partial t}\right] = -i\omega\tilde{f}$
2. $\mathcal{F}[\nabla f] = i\boldsymbol{k}\tilde{f}$