T-61.3010 DSP Table of formulas, spring 2006

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Even and odd functions:
$\mathcal{E}ven\{x(t)\} = 0.5 \cdot [x(t) + x(-t)]$
$\mathcal{O}dd\{x(t)\} = 0.5 \cdot [x(t) - x(-t)]$

Roots of second-order polynomial:
$ax^2 + bx + c = 0$
$x = (-b \pm \sqrt{b^2 - 4ac})/(2a)$

Complex numbers:
$i \equiv j = \sqrt{-1} = -\frac{1}{j}$
$z = x + jy = r\, e^{j\theta}$
$r = \sqrt{x^2 + y^2},\ \theta = \arctan(y/x)$
$x = r\cos(\theta),\ y = r\sin(\theta)$
$e^{j\theta} = \cos(\theta) + j\,\sin(\theta)$
$\sqrt[N]{z} = \sqrt[N]{r\, e^{j\theta}} = |\sqrt[N]{r}|\, e^{j(\theta + 2\pi k)/N},\ k = 0, 1, 2, \ldots, N-1$

Trigonometric functions:
$\mathrm{sinc}(\theta) = \sin(\pi\theta)/(\pi\theta)$
$\sin(\theta)/\theta \to 1$, when $\theta \to 0$; $\mathrm{sinc}(\theta) \to 1$, when $\theta \to 0$
$\cos(\theta) = 0.5e^{j\theta} + 0.5e^{-j\theta}$
$\sin(\theta) = 0.5je^{-j\theta} - 0.5je^{j\theta}$
$\cos^2(\theta) + \sin^2(\theta) = 1$

θ	0	$\pi/6$	$\pi/4$	$\pi/3$
$\sin(\theta)$	0	0.5	$\sqrt{2}/2$	$\sqrt{3}/2$
$\cos(\theta)$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	0.5
θ	$\pi/2$	$3\pi/4$	π	$-\pi/2$
$\sin(\theta)$	1	$\sqrt{2}/2$	0	-1
$\cos(\theta)$	0	$-\sqrt{2}/2$	-1	0

$\pi \approx 3.1416$, $\sqrt{3}/2 \approx 0.8660$, $\sqrt{2}/2 \approx 0.7071$

Geometric series:
$\sum_{n=0}^{+\infty} a^n = \frac{1}{1-a}, \quad |a| < 1$
$\sum_{n=0}^{N} a^n = \frac{1 - a^{N+1}}{1-a}, \quad |a| < 1$

Continuous-time unit step and unit impulse functions:
$\mu(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$
$\delta_\Delta(t) = \frac{d}{dt}\mu_\Delta(t), \quad \delta(t) = \lim_{\Delta \to 0}\delta_\Delta(t)$
$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$
$\int_{-\infty}^{\infty} \delta(t - t_0)x(t)\,dt = x(t_0)$
In DSP notation $2\pi\delta(t)$ is computed $2\pi \int \delta(t) \cdot 1\,dt = 2\pi$, when $t = 0$, and $= 0$ elsewhere.

Discrete-time unit impulse and unit step functions:
$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$
$\mu[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

Convolution
Convolution is commutative, associative and distributive.
$y(t) = h(t) \circledast x(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)\,d\tau$
$y[n] = h[n] \circledast x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$

Correlation:
$r_{xy}[l] = \sum_{n=-\infty}^{+\infty} x[n]y[n-l] = x[l] \circledast y[-l]$
$r_{xx}[l] = \sum_{n=-\infty}^{+\infty} x[n]x[n-l]$

Mean and variance of random signal:
$m_X = \mathrm{E}[X] = \int x p_X(x)dx$
$\sigma_X^2 = \int (x - m_X)^2 p_X(x)dx = \mathrm{E}[X^2] - m_X^2$

Frequencies, angular frequencies, periods:
Here f_s is sampling frequency (also f_T later)
Frequency
f, $[f] = \mathrm{Hz} = 1/\mathrm{s}$
Angular frequency
$\Omega = 2\pi f = 2\pi/T$, $[\Omega] = \mathrm{rad/s}$
Normalized angular frequency
$\omega = 2\pi\Omega/\Omega_s = 2\pi f/f_s$, $[\omega] = \mathrm{rad/sample}$
Normalized frequency in Matlab
$f_{MATLAB} = 2f/f_s$, $[f_{MATLAB}] = 1/\mathrm{sample}$

Integral transform properties
Here all integral transforms share some basic properties. Examples given with CTFT, $x[n] \leftrightarrow X(e^{j\omega})$, $x_1[n] \leftrightarrow X_1(e^{j\omega})$, and $x_2[n] \leftrightarrow X_2(e^{j\omega})$ are time-domain signals with corresponding transform-domain spectra. a and b are constants.
Linearity. All transforms are linear.
$ax_1[n] + bx_2[n] \leftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$
Time-shifting. There is a kernel term in transform, e.g.,
$x[n-k] \leftrightarrow e^{-jk\omega}X(e^{j\omega})$
Frequency-shifting. There is a kernel term in signal e.g.,
$e^{j\omega_k n}x[n] \leftrightarrow X(e^{j(\omega - \omega_k)})$
Conjugate symmetry.
$x^*[n] \leftrightarrow X^*(e^{-j\omega})$
If $x[n] \in \mathbb{R}$, then
$X(e^{j\omega}) = X^*(e^{-j\omega})$
$|X(e^{j\omega})| = |X(e^{-j\omega})|$
$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$
If $x[n] \in \mathbb{R}$ and even, then $X(e^{j\omega}) \in \mathbb{R}$ and even.
If $x[n] \in \mathbb{R}$ and odd, then $X(e^{j\omega})$ purely $\in \mathbb{C}$ and odd.
Time reversal. Transform variable is reversed, e.g.,
$x[-n] \leftrightarrow X(e^{-j\omega})$
Differentiation. In time and frequency domain, e.g.,
$x[n] - x[n-1] \leftrightarrow (1 - e^{-j\omega})X(e^{j\omega})$
$nx[n] \leftrightarrow j\frac{d}{d\omega}X(e^{j\omega})$
Duality. Convolution property: convolution in time domain corresponds multiplication in transform domain
$x_1[n] \circledast x_2[n] \leftrightarrow X_1(e^{j\omega}) \cdot X_2(e^{j\omega})$
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$x_1[n] \cdot x_2 \leftrightarrow \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega - \theta)})\,d\theta$
Parseval's relation. Energy in signal and spectral components:
$\sum |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2\,d\omega$

Integral transforms
Definitions given in first two lines of each type. Some common pairs as well as properties are listed. See math reference book for complete tables.

Fourier-series of continuous-time periodic signals:
$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\Omega_0 t}$
$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\Omega_0 t}\,dt$
$x(t - t_0) \leftrightarrow a_k e^{jk\Omega_0 t_0}$
$e^{jM\Omega_0 t}x(t) \leftrightarrow a_{k-M}$
$\int_T x_a(\tau)x_b(t - \tau)\,d\tau \leftrightarrow Ta_k b_k$
$x_a(t)x_b(t) \leftrightarrow \sum_l a_l b_{k-l}$
$\frac{d}{dt}x(t) \leftrightarrow jk\Omega_0 a_k$

Fourier-series of discrete-time periodic sequences:
$x[n] = \sum_{k=\langle N\rangle} a_k\, e^{jk\omega_0 n}, \quad x[n]$ periodic with N_0
$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\omega_0 n}, \quad a_k$ periodic with N_0
$x[n - M] \leftrightarrow a_k e^{jk\omega_0 M}$

$e^{jM\omega_0 n}x[n] \leftrightarrow a_{k-M}$

Continuous-time Fourier-transform (CTFT):

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\,\mathrm{d}\Omega$
$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\,\mathrm{d}t$
$x(t-t_k) \leftrightarrow e^{j\Omega t_k}X(j\Omega)$
$e^{j\Omega_k t}x(t) \leftrightarrow X(j(\Omega-\Omega_k))$
$x_a(t) \circledast x_b(t) \leftrightarrow X_a(j\Omega)X_b(j\Omega)$
$x_a(t)x_b(t) \leftrightarrow \frac{1}{2\pi}X_a(j\Omega) \circledast X_b(j\Omega)$
$\frac{\mathrm{d}}{\mathrm{d}t}x(t) \leftrightarrow j\Omega X(j\Omega)$
$tx(t) \leftrightarrow j\frac{\mathrm{d}}{\mathrm{d}\Omega}X(j\Omega)$
$e^{j\Omega_0 t} \leftrightarrow 2\pi\delta(\Omega-\Omega_0)$
$\cos(\Omega_0 t) \leftrightarrow \pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)]$
$\sin(\Omega_0 t) \leftrightarrow j\pi[\delta(\Omega+\Omega_0)-\delta(\Omega-\Omega_0)]$
$x(t)=1 \leftrightarrow 2\pi\delta(\Omega)$
$x(t) = \begin{cases}1, |t|<T_1\\ 0, |t|>T_1\end{cases} \leftrightarrow \frac{2\sin(\Omega T_1)}{\Omega}$
$\frac{\sin(Wt)}{\pi t} \leftrightarrow X(j\Omega) = \begin{cases}1, |\Omega|<W\\ 0, |\Omega|>W\end{cases}$
$\delta(t) \leftrightarrow 1$
$\delta(t-t_k) \leftrightarrow e^{j\Omega t_k}$
$e^{-at}\mu(t) \leftrightarrow \frac{1}{a+j\Omega}$, where $\mathcal{R}eal\{a\}>0$

Discrete-time Fourier-transform (DTFT):

$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\,\mathrm{d}\omega$
$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$, $X(e^{j\omega})$ periodic 2π
$x[n-k] \leftrightarrow e^{-jk\omega}X(e^{j\omega})$
$e^{j\omega_k n}x[n] \leftrightarrow X(e^{j(\omega-\omega_k)})$
$x_1[n] \circledast x_2[n] \leftrightarrow X_1(e^{j\omega})\cdot X_2(e^{j\omega})$
$x_1[n]\cdot x_2 \leftrightarrow \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})\,\mathrm{d}\theta$
$nx[n] \leftrightarrow j\frac{\mathrm{d}}{\mathrm{d}\omega}X(e^{j\omega})$
$e^{j\omega_0 n} \leftrightarrow 2\pi\sum_l \delta(\omega-\omega_0-2\pi l)$
$\cos(\omega_0 n) \leftrightarrow \pi\sum_l[\delta(\omega-\omega_0-2\pi l)+\delta(\omega+\omega_0-2\pi l)$
$\sin(\omega_0 n) \leftrightarrow j\pi\sum_l[\delta(\omega+\omega_0-2\pi l)-\delta(\omega-\omega_0-2\pi l)$
$x[n]=1 \leftrightarrow 2\pi\sum_l \delta(\omega-2\pi l)$
$x[n] = \begin{cases}1, |n|\le N_1\\ 0, |n|>N_1\end{cases} \leftrightarrow \frac{\sin(\omega(N_1+0.5))}{\sin(\omega/2)}$
$\frac{\sin(Wn)}{\pi n} = \frac{W}{\pi}\mathrm{sinc}(\frac{Wn}{\pi}) \leftrightarrow \ldots$
$\ldots X(e^{j\omega}) = \begin{cases}1, & 0\le|\omega|\le W\\ 0, & W<|\omega|\le\pi\end{cases}$
$\delta[n] \leftrightarrow 1$
$\delta[n-k] \leftrightarrow e^{-jk\omega}$
$a^n\mu[n] \leftrightarrow \frac{1}{1-ae^{-j\omega}}$, $|a|<1$

Discrete Fourier-transform (DFT):

Connection to DTFT: $X[k] = X(e^{j\omega})|_{\omega=2\pi k/N}$
$W_N = e^{-j2\pi/N}$
$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}$, $0\le k\le N-1$
$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}$, $0\le k\le N-1$

Laplace transform:

Convergence with a certain ROC (region of convergence).
Connection to continuous-time Fourier-transform: $s=j\Omega$
$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,\mathrm{d}s$
$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\,\mathrm{d}t$

z-transform:

Convergence with a certain ROC (region of convergence).
Connection to discrete-time Fourier-transform: $z=e^{j\omega}$
$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}\mathrm{d}z$
$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$
$x[n-k] \leftrightarrow z^{-k}X(z)$
$x_1[n] \circledast x_2[n] \leftrightarrow X_1(z)\cdot X_2(z)$

$\delta[n] \leftrightarrow 1$, ROC all z
$\delta[n-k] \leftrightarrow z^{-k}$, all z, except 0 $(k>0)$ or ∞ $(k<0)$
$\mu[n] \leftrightarrow \frac{1}{1-z^{-1}}$, $|z|>1$
$-\mu[-n-1] \leftrightarrow \frac{1}{1-z^{-1}}$, $|z|<1$
$a^n\mu[n] \leftrightarrow \frac{1}{1-az^{-1}}$, $|z|>|a|$
$na^n\mu[n] \leftrightarrow \frac{az^{-1}}{(1-az^{-1})^2}$, $|z|>|a|$
$(n+1)a^n\mu[n] \leftrightarrow \frac{1}{(1-az^{-1})^2}$, $|z|>|a|$
$r^n\cos(\omega_0 n)\mu[n] \leftrightarrow \frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$, $|z|>|r|$
$r^n\sin(\omega_0 n)\mu[n] \leftrightarrow \frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$, $|z|>|r|$

LTI filter analysis

Stability $\sum_h |h[n]|<\infty$; unit cirle belongs to ROC
Causality $h[n]=0, n<0$; ∞ belongs to ROC
Unit step response $s[n]=\sum_{k=-\infty}^{n} h[k]$
Causal transfer function of order $\max\{M,N\}$:
$H(z) = K\cdot\frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{n=0}^{N} a_n z^{-n}} = G\cdot\frac{\prod_{m=1}^{M}(1-d_m z^{-1})}{\prod_{n=1}^{N}(1-p_n z^{-1})}$
Zeros: $B(z)=0$; *Poles*: $A(z)=0$; where $H(z)=B(z)/A(z)$
Frequency, magnitude/amplitude, phase response, $z\leftarrow e^{j\omega}$
$H(e^{j\omega}) = |H(e^{j\omega})|\, e^{j\angle H(e^{j\omega})}$
$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$
$H[k] = H(e^{j\omega})|_{\omega=2\pi k/N}$
Group delay $\tau(\omega) = -\frac{\mathrm{d}}{\mathrm{d}\omega}\angle H(e^{j\omega})$
Important *transform pairs and properties*:
$a\,\delta[n-k] \leftrightarrow a\,e^{-jk\omega} \leftrightarrow a\,z^{-k}$
$a^n\mu[n] \leftrightarrow 1/[1-a\,e^{-j\omega}] \leftrightarrow 1/[1-a\,z^{-1}]$
$a\,x[n-k] \leftrightarrow a\,e^{-jk\omega}X(e^{j\omega}) \leftrightarrow a\,z^{-k}X(z)$
$y[n]=h[n]\circledast x[n] \leftrightarrow Y(z)=H(z)\cdot X(z)$
rectangular $\leftrightarrow$ sinc, sinc $\leftrightarrow$ rectangular

LTI filter design (synthesis)

Bilinear transform $H(z)=H(s)|_s$ and *prewarping*
$s = k\cdot(1-z^{-1})/(1+z^{-1})$, $k=1$ or $k=2/T=2f_T$
$\Omega_{prewarp,c} = k\cdot\tan(\omega_c/2)$, $k=1$ or $k=2/T=2f_T$
Spectral transformations, $\hat{\omega}_c$ desired cut-off
LP-LP $z^{-1} = (\hat{z}^{-1}-\alpha)/(1-\alpha\hat{z}^{-1})$, where
$\alpha = \sin(0.5(\omega_c-\hat{\omega}_c))/\sin(0.5(\omega_c+\hat{\omega}_c))$
LP-HP $z^{-1} = -(\hat{z}^{-1}+\alpha)/(1+\alpha\hat{z}^{-1})$, where
$\alpha = -\cos(0.5(\omega_c+\hat{\omega}_c))/\cos(0.5(\omega_c-\hat{\omega}_c))$
Windowed Fourier series method
$H(e^{j\omega}) = \begin{cases}1, & |\omega|<\omega_c\\ 0, & |\omega|\ge\omega_c\end{cases} \leftrightarrow \ldots$
$\ldots h[n] = \frac{\sin(\omega_c n)}{\pi n} = \frac{\omega_c}{\pi}\mathrm{sinc}(\frac{\omega_c n}{\pi})$
$h_{FIR}[n] = h_{ideal}[n]\cdot w[n]$
$H_{FIR}(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{ideal}(e^{j\theta})W(e^{j(\omega-\theta)})\,\mathrm{d}\theta$
Fixed window functions, order $N=2M$, $-M\le n\le M$:
Rectangular $w[n]=1$
Hamming $w[n]=0.54+0.46\cos((2\pi n)/(2M))$
Hann $w[n]=0.5\cdot(1+\cos((2\pi n)/(2M)))$
Blackman $w[n] = 0.42 + 0.5\cos((2\pi n)/(2M)) + 0.08\cos((4\pi n)/(2M))$
Bartlett $w[n]=1-(|n|/M)$

Multirate systems

Upsampling with factor L, $\boxed{\uparrow L}$
$x_u[n] = \begin{cases}x[n/L], n=0,\pm L,\pm 2L,\ldots\\ x_u[n]=0, \text{ otherwise}\end{cases}$
$X_u(z)=X(z^L)$, $X_u(e^{j\omega})=X(e^{j\omega L})$
Downsampling with factor M, $\boxed{\downarrow M}$
$x_d[n]=x[nM]$
$X_d(z) = (1/M)\sum_{k=0}^{M-1} X(z^{1/M}W_M^{-k})$,
$X_d(e^{j\omega}) = (1/M)\sum_{k=0}^{M-1} X(e^{j(\omega-2\pi k)/M})$

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$r = \sqrt{x^2+y^2},\ \theta = \arctan(y/x)$
$x = r\cos(\theta),\ y = r\sin(\theta)$
$e^{j\theta} = \cos(\theta) + j\,\sin(\theta)$
$\sqrt[N]{z} = \sqrt[N]{r\,e^{j\theta}} = |\sqrt[N]{r}|\,e^{j(\theta+2\pi k)/N},\ k = 0, 1, 2, \ldots, N-1$

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$$\mu(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

$\delta_\Delta(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mu_\Delta(t),\quad \delta(t) = \lim_{\Delta\to 0}\delta_\Delta(t)$
$\int_{-\infty}^{\infty}\delta(t)\,\mathrm{d}t = 1$
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Discrete-time unit impulse and unit step functions:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

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$x(t-t_0) \leftrightarrow a_k e^{jk\Omega_0 t_0}$
$e^{jM\Omega_0 t}x(t) \leftrightarrow a_{k-M}$
$\int_T x_a(\tau)x_b(t-\tau)\,\mathrm{d}\tau \leftrightarrow Ta_k b_k$
$x_a(t)x_b(t) \leftrightarrow \sum_l a_l b_{k-l}$
$\frac{\mathrm{d}}{\mathrm{d}t}x(t) \leftrightarrow jk\Omega_0 a_k$

Fourier-series of discrete-time periodic sequences:

$x[n] = \sum_{k=\langle N\rangle} a_k\,e^{jk\omega_0 n},\quad x[n]$ periodic with N_0
$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\,e^{-jk\omega_0 n},\quad a_k$ periodic with N_0
$x[n-M] \leftrightarrow a_k e^{jk\omega_0 M}$

$e^{jM\omega_0 n}x[n] \leftrightarrow a_{k-M}$

Continuous-time Fourier-transform (CTFT):

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, \mathrm{d}\Omega$

$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, \mathrm{d}t$

$x(t-t_k) \leftrightarrow e^{j\Omega t_k} X(j\Omega)$

$e^{j\Omega_k t}x(t) \leftrightarrow X(j(\Omega-\Omega_k))$

$x_a(t) \circledast x_b(t) \leftrightarrow X_a(j\Omega)X_b(j\Omega)$

$x_a(t)x_b(t) \leftrightarrow \frac{1}{2\pi} X_a(j\Omega) \circledast X_b(j\Omega)$

$\frac{\mathrm{d}}{\mathrm{d}t}x(t) \leftrightarrow j\Omega X(j\Omega)$

$tx(t) \leftrightarrow j\frac{\mathrm{d}}{\mathrm{d}\Omega}X(j\Omega)$

$e^{j\Omega_0 t} \leftrightarrow 2\pi\delta(\Omega-\Omega_0)$

$\cos(\Omega_0 t) \leftrightarrow \pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)]$

$\sin(\Omega_0 t) \leftrightarrow j\pi[\delta(\Omega+\Omega_0)-\delta(\Omega-\Omega_0)]$

$x(t) = 1 \leftrightarrow 2\pi\delta(\Omega)$

$x(t) = \begin{cases} 1, |t| < T_1 \\ 0, |t| > T_1 \end{cases} \quad \leftrightarrow \frac{2\sin(\Omega T_1)}{\Omega}$

$\frac{\sin(Wt)}{\pi t} \leftrightarrow X(j\Omega) = \begin{cases} 1, |\Omega| < W \\ 0, |\Omega| > W \end{cases}$

$\delta(t) \leftrightarrow 1$

$\delta(t-t_k) \leftrightarrow e^{j\Omega t_k}$

$e^{-at}\mu(t) \leftrightarrow \frac{1}{a+j\Omega}$, where $\mathcal{R}eal\{a\} > 0$

Discrete-time Fourier-transform (DTFT):

$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, \mathrm{d}\omega$

$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \quad X(e^{j\omega})$ periodic 2π

$x[n-k] \leftrightarrow e^{-jk\omega}X(e^{j\omega})$

$e^{j\omega_k n}x[n] \leftrightarrow X(e^{j(\omega-\omega_k)})$

$x_1[n] \circledast x_2[n] \leftrightarrow X_1(e^{j\omega}) \cdot X_2(e^{j\omega})$

$x_1[n] \cdot x_2 \leftrightarrow \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})\, \mathrm{d}\theta$

$nx[n] \leftrightarrow j\frac{\mathrm{d}}{\mathrm{d}\omega}X(e^{j\omega})$

$e^{j\omega_0 n} \leftrightarrow 2\pi\sum_l \delta(\omega-\omega_0-2\pi l)$

$\cos(\omega_0 n) \leftrightarrow \pi\sum_l[\delta(\omega-\omega_0-2\pi l)+\delta(\omega+\omega_0-2\pi l)$

$\sin(\omega_0 n) \leftrightarrow j\pi\sum_l[\delta(\omega+\omega_0-2\pi l)-\delta(\omega-\omega_0-2\pi l)$

$x[n] = 1 \leftrightarrow 2\pi\sum_l \delta(\omega-2\pi l)$

$x[n] = \begin{cases} 1, |n| \le N_1 \\ 0, |n| > N_1 \end{cases} \quad \leftrightarrow \frac{\sin(\omega(N_1+0.5))}{\sin(\omega/2)}$

$\frac{\sin(Wn)}{\pi n} = \frac{W}{\pi}\mathrm{sinc}(\frac{Wn}{\pi}) \leftrightarrow \ldots$

$\ldots X(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le W \\ 0, & W < |\omega| \le \pi \end{cases}$

$\delta[n] \leftrightarrow 1$

$\delta[n-k] \leftrightarrow e^{-jk\omega}$

$a^n\mu[n] \leftrightarrow \frac{1}{1-ae^{-j\omega}}, \quad |a| < 1$

Discrete Fourier-transform (DFT):

Connection to DTFT: $X[k] = X(e^{j\omega})|_{\omega=2\pi k/N}$

$W_N = e^{-j2\pi/N}$

$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}, \quad 0 \le k \le N-1$

$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}, \quad 0 \le k \le N-1$

Laplace transform:

Convergence with a certain ROC (region of convergence).
Connection to continuous-time Fourier-transform: $s = j\Omega$

$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\, \mathrm{d}s$

$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, \mathrm{d}t$

z-transform:

Convergence with a certain ROC (region of convergence).
Connection to discrete-time Fourier-transform: $z = e^{j\omega}$

$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}\mathrm{d}z$

$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$

$x[n-k] \leftrightarrow z^{-k}X(z)$

$x_1[n] \circledast x_2[n] \leftrightarrow X_1(z) \cdot X_2(z)$

$\delta[n] \leftrightarrow 1, \quad$ ROC all z

$\delta[n-k] \leftrightarrow z^{-k}, \quad$ all z, except 0 $(k > 0)$ or ∞ $(k < 0)$

$\mu[n] \leftrightarrow \frac{1}{1-z^{-1}}, \quad |z| > 1$

$-\mu[-n-1] \leftrightarrow \frac{1}{1-z^{-1}}, \quad |z| < 1$

$a^n\mu[n] \leftrightarrow \frac{1}{1-az^{-1}}, \quad |z| > |a|$

$na^n\mu[n] \leftrightarrow \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| > |a|$

$(n+1)a^n\mu[n] \leftrightarrow \frac{1}{(1-az^{-1})^2}, \quad |z| > |a|$

$r^n\cos(\omega_0 n)\mu[n] \leftrightarrow \frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}, \quad |z| > |r|$

$r^n\sin(\omega_0 n)\mu[n] \leftrightarrow \frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}, \quad |z| > |r|$

LTI filter analysis

Stability $\sum_h |h[n]| < \infty$; unit cirle belongs to ROC

Causality $h[n] = 0, n < 0$; ∞ belongs to ROC

Unit step response $s[n] = \sum_{k=-\infty}^{n} h[k]$

Causal transfer function of order $\max\{M, N\}$:

$H(z) = K \cdot \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{n=0}^{N} a_n z^{-n}} = G \cdot \frac{\prod_{m=1}^{M}(1-d_m z^{-1})}{\prod_{n=1}^{N}(1-p_n z^{-1})}$

Zeros: $B(z) = 0$; *Poles*: $A(z) = 0$; where $H(z) = B(z)/A(z)$

Frequency, magnitude/amplitude, phase response, $z \leftarrow e^{j\omega}$

$H(e^{j\omega}) = |H(e^{j\omega})|\, e^{j\angle H(e^{j\omega})}$

$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$

$H[k] = H(e^{j\omega})|_{\omega=2\pi k/N}$

Group delay $\tau(\omega) = -\frac{\mathrm{d}}{\mathrm{d}\omega}\angle H(e^{j\omega})$

Important *transform pairs and properties*:

$a\,\delta[n-k] \leftrightarrow a\, e^{-jk\omega} \leftrightarrow a\, z^{-k}$

$a^n\mu[n] \leftrightarrow 1/[1-a\, e^{-j\omega}] \leftrightarrow 1/[1-a\, z^{-1}]$

$a\, x[n-k] \leftrightarrow a\, e^{-jk\omega}X(e^{j\omega}) \leftrightarrow a\, z^{-k}X(z)$

$y[n] = h[n] \circledast x[n] \leftrightarrow Y(z) = H(z) \cdot X(z)$

rectangular $\leftrightarrow$ sinc, sinc $\leftrightarrow$ rectangular

LTI filter design (synthesis)

Bilinear transform $H(z) = H(s)|_s$ and *prewarping*

$s = k \cdot (1-z^{-1})/(1+z^{-1}), \qquad k = 1$ or $k = 2/T = 2f_T$

$\Omega_{prewarp,c} = k \cdot \tan(\omega_c/2), \qquad k = 1$ or $k = 2/T = 2f_T$

Spectral transformations, $\hat{\omega}_c$ desired cut-off

LP-LP $z^{-1} = (\hat{z}^{-1}-\alpha)/(1-\alpha\hat{z}^{-1})$, where

$\alpha = \sin(0.5(\omega_c-\hat{\omega}_c))/\sin(0.5(\omega_c+\hat{\omega}_c))$

LP-HP $z^{-1} = -(\hat{z}^{-1}+\alpha)/(1+\alpha\hat{z}^{-1})$, where

$\alpha = -\cos(0.5(\omega_c+\hat{\omega}_c))/\cos(0.5(\omega_c-\hat{\omega}_c))$

Windowed Fourier series method

$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| \ge \omega_c \end{cases} \quad \leftrightarrow \ldots$

$\ldots h[n] = \frac{\sin(\omega_c n)}{\pi n} = \frac{\omega_c}{\pi}\,\mathrm{sinc}(\frac{\omega_c n}{\pi})$

$h_{FIR}[n] = h_{ideal}[n] \cdot w[n]$

$H_{FIR}(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{ideal}(e^{j\theta})W(e^{j(\omega-\theta)})\, \mathrm{d}\theta$

Fixed window functions, order $N = 2M$, $-M \le n \le M$:

Rectangular $w[n] = 1$

Hamming $w[n] = 0.54 + 0.46\cos((2\pi n)/(2M))$

Hann $w[n] = 0.5 \cdot (1+\cos((2\pi n)/(2M)))$

Blackman $w[n] = 0.42 + 0.5\cos((2\pi n)/(2M)) + 0.08\cos((4\pi n)/(2M))$

Bartlett $w[n] = 1 - (|n|/M)$

Multirate systems

Upsampling with factor L, $\boxed{\uparrow L}$

$x_u[n] = \begin{cases} x[n/L], n = 0, \pm L, \pm 2L, \ldots \\ x_u[n] = 0, \text{ otherwise} \end{cases}$

$X_u(z) = X(z^L)$, $X_u(e^{j\omega}) = X(e^{j\omega L})$

Downsampling with factor M, $\boxed{\downarrow M}$

$x_d[n] = x[nM]$

$X_d(z) = (1/M)\sum_{k=0}^{M-1} X(z^{1/M}W_M^{-k})$,

$X_d(e^{j\omega}) = (1/M)\sum_{k=0}^{M-1} X(e^{j(\omega-2\pi k)/M})$

Muista täyttää kurssin palautekysely osoitteessa
http://www.cs.hut.fi/Opinnot/Palaute/kurssipalaute.html

T-61.3010 DIGITAALINEN SIGNAALINKÄSITTELY JA SUODATUS

Tentti / 18.12.2006 / OS

1. (a) Mikä on sekvenssin $x[n] = e^{j(3\pi n/7)}$ jakso? (2p)

   (b) Laske sekvenssien $h[n] = \delta[n] - 2\delta[n-1] + \delta[n-2]$ ja $x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]$ konvoluutio, $y[n] = h[n] * x[n]$. Minkälaista suodatinta (alipäästö, ylipäästö, kaistanpäästö tai kaistanesto) $h[n]$:n kuvaama impulssivaste edustaa? (2p)

2. (a) Tarkastellaan digitaalisuodattimen siirtofunktiota $H(z) = \dfrac{1 - z^{-2}}{1 + 0.5z^{-1} - 0.5z^{-2}}$.
   Tutki, onko suodatin stabiili ja kausaalinen. Perustele vastauksesi! Piirrä suodattimen toteutusta kuvaava virtauskaavio mahdollisimman yksinkertaisessa muodossa. (2p)

   (b) Laske suodattimen $H(z) = \frac{1}{2}(1 + z^{-4})$ taajuusvaste. Hahmottele suodattimen amplitudivaste $|H(e^{j\omega})|$. (2p)

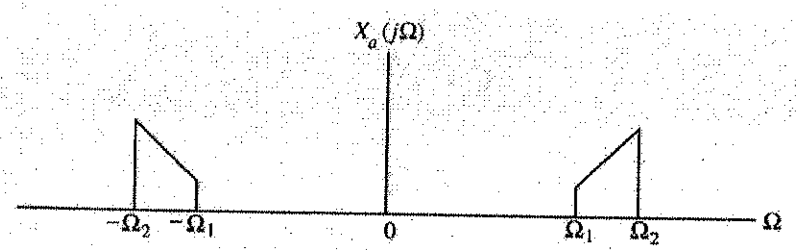
3. Tarkastellaan kahta äärellisen impulssivasteen (FIR) systeemiä, joiden impulssivasteet ovat

$$h_1[n] = \delta[n] + 2\delta[n-2] + \delta[n-4]$$
$$h_2[n] = \delta[n] - \delta[n-4]$$

   (a) Muodosta systeemien $h_1[n]$ ja $h_2[n]$ kaskadikytkennän impulssivaste $h[n]$ ja siirtofunktio $H(z)$.
   (b) Laske kaskadikytkennän taajuusvasteen itseisarvo ja vaihe sekä hahmottele näiden kuvaajat.
   (c) Määritä kaskadikytkennän askelvaste. Miten askelvaste käyttäytyy, kun n on suuri?
   (d) Miten systeemien $h_1[n]$ ja $h_2[n]$ rinnankytkennän vaihe käyttäytyy?

   (6 p)

4. Tarkastellaan ns. kaistanpäästösignaalia (bandpass signal) $x_a(t)$, joka on kaistarajoitettu tietylle taajuuskaistalle $\Omega_1 \le |\Omega| \le \Omega_2$, missä $\Omega_1 > 0$. Kaistanpäästösignaalin $x_a(t)$ spektri $X_a(j\Omega)$ on esitetty alla.

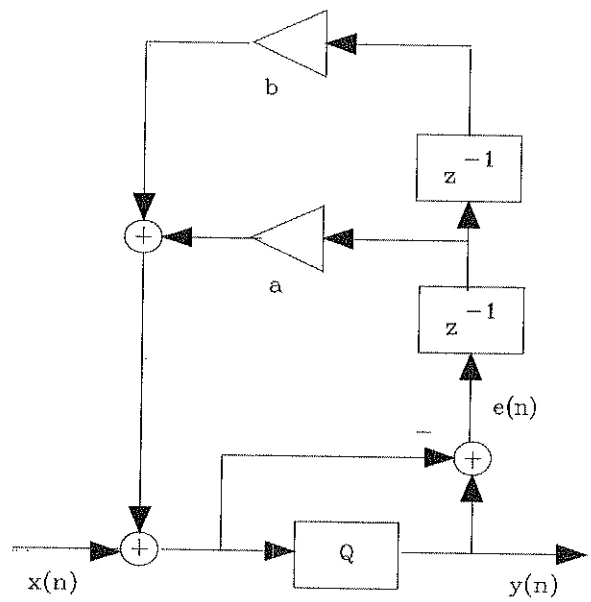


Määrää pienin mahdollinen näytteenottotaajuus F_T, jota voidaan käyttää kaistanpäästösignaalin $x_a(t)$ näytteistämiseen siten, että $x_a(t)$ voidaan täydellisesti palauttaa syntyneestä diskreetistä signaalista $x[n]$ molemmissa alla esitettyjen (a)- ja (b)- kohdan rajataajuuksien Ω_1 ja Ω_2 tapauksissa. Hahmottele näytteenotolla signaalista $x_a(t)$ muodostetun diskreetin sigaalin $x[n]$ spektri $X_p(j\Omega)$ molemmissa tapauksissa, (a) ja (b), pienimmällä mahdollisella näytteenotto-taajuudella F_T.

   (a) $\Omega_1 = 160\pi$ ja $\Omega_2 = 200\pi$; (2p)
   (b) $\Omega_1 = 110\pi$ ja $\Omega_2 = 150\pi$; (2p)

KÄÄNNÄ!

5. Kvantisointivirhettä voidaan kompensoida ns. virheen takaisinkytkennän (error feedback) avulla. Menetelmässä suodatettu virhesignaali lisätään kvantisointia (Q[.]) edeltävään haaraan suodatinrakenteessa. Ilman takaisinkytkentää virhesignaali $e[n]$ systeemissä on puhdas kvantisointivirhe, ts. $e[n]=y[n]-x[n]$; kompensoidussa piirissä virhesignaali on lähdön $y[n]$ ja kompensoidun tulosignaalin välinen erotus. Oheisessa kuvassa on toisen asteen error feedback -rakenne.



(a) Määrää rakenteen kohinasiirtofunktio $H_e(z)$

$$E_{tot}(z) = H_e(z)\, E(z)\,,$$

missä $E(z)$ on virheen $e[n] = Q[x[n]] - x[n]$ z-muunnos ja $E_{tot}(z)$ kokonaisvirheen $e_{tot}[n] = y[n] - x[n]$ z-muunnos.

(b) Määrää siirtofunktion $H_e(z)$ amplitudivaste, kun $a = 2$ ja $b = 1$.
Hahmottele amplitudivasteen kuvaaja. Miten kohinan spektri muuttuu?

(c) Mitä tapahtuu virheen varianssille?

(6 p)

5. Kehitä rekursiivinen algoritmi, joka generoi jonon n^3 (0,1,8,27,...). Algoritmi on muotoa

$$y[n] = \sum_{i=1}^{N} a_i y[n-i] + b$$

missä a_i ja b ovat vakioita.
Mitkä ovat tarvittavat alkuarvot?

Vihjeitä: Tutki sekvenssin peräkkäisiä termejä esimerkiksi lausekkeiden $(n-1)^3$, n^3, $(n+1)^3$ avulla ja lausu ne $y[n]$:n sekä sen siirrettyjen esiintymien avulla.

Toinen mahdollisuus on tarkastella z-muunnoksen avulla rekursiivisen generaattorin siirtofunktiota. Generaattorissa ei ole input-signaalia, mutta se voidaan ajatella "käynnistetyksi" yksikköimpulssiherätteellä.

$$Z\{n^3\} = \frac{z(z^2 + 4z + 1)}{(z-1)^4}$$

(6 p)

Remember to fill in the course feedback form
http://www.cs.hut.fi/Opinnot/Palaute/kurssipalaute-en.html

T-61.3010 DIGITAL SIGNAL PROCESSING AND FILTERING

Examination / 18.12.2006 / OS

1. (a) What is the period of the sequence $x[n] = e^{j(3\pi n/7)}$? (2p)

   (b) Calculate the convolution, $y[n] = h[n] * x[n]$, of the sequences $h[n] = \delta[n] - 2\delta[n-1] + \delta[n-2]$ and $x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]$.
   What kind of filter (lowpass, highpass, bandpass, or bandstop) the impulse response $h[n]$ represents?

   (2p)

2. (a) Consider the digital filter transfer function $H(z) = \dfrac{1 - z^{-2}}{1 + 0.5z^{-1} - 0.5z^{-2}}$.
   Is the filter stable and causal? Justify your answer! Draw the signal flow diagram describing the computations of the filter in its simplest form. (2p)

   (b) Calculate the frequency response of the filter $H(z) = \frac{1}{2}(1 + z^{-4})$. Sketch the amplitude response of the filter, i.e., $|H(e^{j\omega})|$. (2p)

3. Consider the finite impulse response (FIR) systems with the unit impulse responses given below

$$h_1[n] = \delta[n] + 2\delta[n-2] + \delta[n-4]$$

$$h_2[n] = \delta[n] - \delta[n-4]$$

   (a) Determine the unit impulse response $h[n]$ and the transfer function $H(z)$ of the cascade connection of $h_1[n]$ and $h_2[n]$.

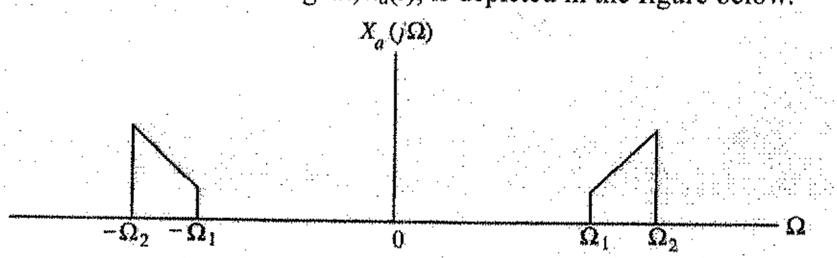
   (b) Calculate the frequency response, i.e., the magnitude and phase responses, of the cascade system and sketch them graphically.

   (c) Determine the step response of the cascade system. How does the step response behave with large values of n.

   (d) How does the phase response of the parallel connection of $h_1[n]$ and $h_2[n]$ behave? Justify your answer!

   (6 p)

4. Consider a bandpass signal which is bandlimited to a range $\Omega_1 \leq |\Omega| \leq \Omega_2$, where $\Omega_1 > 0$. The spectrum, $X_a(j\Omega)$, of a continuous-time bandlimited signal, $x_a(t)$, is depicted in the figure below.



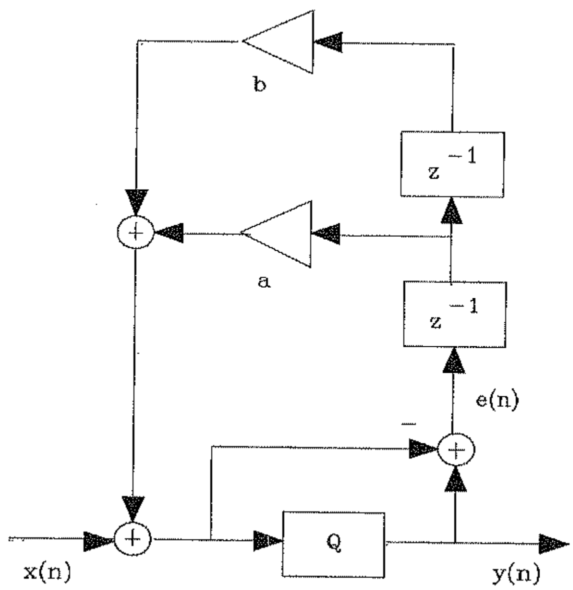
Determine the smallest sampling rate F_T that can be employed to sample the bandpass signal $x_a(t)$ so that it can be fully recovered from its samples, $x[n]$, for both of the following sets, (a) and (b), of bandedges Ω_1 and Ω_2. Sketch the Fourier transform, $X_p(j\Omega)$, of the sampled version $x[n]$ obtained by sampling $x_a(t)$ at the smallest possible sampling rate F_T in both cases (a) and (b).

   (a) $\Omega_1 = 160\pi$ and $\Omega_2 = 200\pi$; (2p)

   (b) $\Omega_1 = 110\pi$ and $\Omega_2 = 150\pi$; (2p)

TURN OVER!

5. Quantization error can be compensated using the so called error-feedback (or error spectrum shaping). In this method, the filtered error signal is added to the signal branch before quantization (Q[.]). Without error-feedback the error signal $e[n]$ in the system is the pure quantization error, i.e., $e[n] = y[n] - x[n]$; in the compensated structure the error signal is the difference between the output $y[n]$ and the compensated input signal. A second order error-feedback structure is depicted below.



(a) Determine the noise transfer function $He(z)$

$$Etot(z) = He(z)\,E(z)\,,$$

where $E(z)$ is the z-transform of the error signal $e[n] = \text{Q}[x[n]] - x[n]$ and $Etot(z)$ is the z-transform of total error $etot[n] = y[n] - x[n]$ in the compensated structure.

(b) Determine the amplitude response of $He(z)$ with $a = 2$ and $b = 1$. Sketch the amplitude response. How the noise spectrum is changed?

(a) What happens to the noise variance?

(6 p)

6. Derive a recursive algorithm that generates the sequence n^3 (0,1,8,27,...). The algorithm is of the form

$$y[n] = \sum_{i=1}^{N} a_i y[n-i] + b$$

where a_i and b are constants.
What are the necessary initial values?

Hints: Investigate the consecutive terms of the sequence, e.g. using expressions $(n\text{-}1)^3$, n^3, $(n\text{+}1)^3$ and express them with y[n] and its shifted occurrences.

Another alternative is to consider the z-domain transfer function of the recursive generator. The generator does not have any input signal, but it can be "started" using a unit impulse input.

$$Z\{n^3\} = \frac{z(z^2+4z+1)}{(z-1)^4}$$

(6 p)