TKK, Matematiikan laitos
Mat-1.1331 peruskurssi KP3 I
Loppukoe

Rasila
syksy 2006
Pe 27.10.2006

1. Oletetaan, että $z = 3 + 2\mathrm{i}$, $w = 1 + \mathrm{i}$. Laske

(a) $(z + \bar{w})^2$,

(b) z/w,

(c) $\mathrm{Arg}(zw)$.

Anna ratkaisusi muodossa $x + \mathrm{i}y$.

2. Etsi funktion

$$u(x, y) = -3x(y + 1)^2 + x^3$$

harmoninen konjugaattifunktio.

3. Oletetaan, että $c > 0$ ja

$$f(t) = \begin{cases} -c, & \text{kun } -\pi \le t < 0, \\ c, & \text{kun } 0 \le t < \pi. \end{cases}$$

Laske funktion $f(t)$ Fourier-sarja (f:n jakso on 2π).

4. Määritä seuraavien funktioiden Laplace-muunnokset:

(a)

$$f(t) = \int_0^t \sin(\tau)\, d\tau,$$

(b)

$$e^{-at}\cos(t + 1).$$

5. Ratkaise Laplace-muuntamalla ja käänteismuuntamalla

$$y'' + 2y' + 5y = 0,$$

alkuarvoilla $y(0) = 1$, $y'(0) = 1$.

Kaavoja:

Hyperboliset ja trigonometriset funktiot

$$\cosh z = \frac{e^z + e^{-z}}{2}, \qquad \sinh z = \frac{e^z - e^{-z}}{2},$$

$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{\cosh z}{\sinh z},$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i},$$

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y),$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

Analyyttinen funktio $f\colon D \to \mathbb{C}$

$$f(z) = u(x, y) + iv(x, y),$$

tällöin Cauchy-Riemannin yhtälöt ovat:

$$\frac{\partial}{\partial x}u(x, y) = \frac{\partial}{\partial y}v(x, y), \qquad \frac{\partial}{\partial y}u(x, y) = -\frac{\partial}{\partial x}v(x, y).$$

T-jaksoisen funktion Fourier-sarja

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right],$$

missä $\omega = 2\pi/T$ ja

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\, dt, \quad a_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(k\omega t)\, dt,$$

ja

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(k\omega t)\, dt.$$

Lisää kaavoja kääntöpuolella.

Laplace Transform:

General Formulas

Formula	Name, Comments
$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Definition of Transform Inverse Transform
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity
$\mathcal{L}\{e^{at} f(t)\} = F(s - a)$ $\mathcal{L}^{-1}\{F(s - a)\} = e^{at} f(t)$	s-Shifting (First Shifting Theorem)
$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{(n-1)}f(0) - \cdots$ $\cdots - f^{(n-1)}(0)$ $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}(f)$	Differentiation of Function Integration of Function
$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)\,d\tau$ $= \int_0^t f(t - \tau)g(\tau)\,d\tau$ $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$	Convolution
$\mathcal{L}\{f(t - a)\,u(t - a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t - a)\,u(t - a)$	t-Shifting (Second Shifting Theorem)
$\mathcal{L}\{tf(t)\} = -F'(s)$ $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\,d\tilde{s}$	Differentiation of Transform Integration of Transform
$\mathcal{L}(f) = \frac{1}{1 - e^{-ps}}\int_0^p e^{-st} f(t)\,dt$	f Periodic with Period p

Laplace Transforms

For more extensive tables, see Ref. [A9] *in Appendix 1.*

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$
1	$1/s$	1
2	$1/s^2$	t
3	$1/s^n \quad (n = 1, 2, \cdots)$	$t^{n-1}/(n - 1)!$
4	$1/\sqrt{s}$	$1/\sqrt{\pi t}$
5	$1/s^{3/2}$	$2\sqrt{t/\pi}$
6	$1/s^a \quad (a > 0)$	$t^{a-1}/\Gamma(a)$
7	$\frac{1}{s - a}$	e^{at}
8	$\frac{1}{(s - a)^2}$	te^{at}
9	$\frac{1}{(s - a)^n} \quad (n = 1, 2, \cdots)$	$\frac{1}{(n - 1)!}t^{n-1}e^{at}$
10	$\frac{1}{(s - a)^k} \quad (k > 0)$	$\frac{1}{\Gamma(k)}t^{k-1}e^{at}$
11	$\frac{1}{(s - a)(s - b)} \quad (a \neq b)$	$\frac{1}{(a - b)}(e^{at} - e^{bt})$
12	$\frac{s}{(s - a)(s - b)} \quad (a \neq b)$	$\frac{1}{(a - b)}(ae^{at} - be^{bt})$
13	$\frac{1}{s^2 + \omega^2}$	$\frac{1}{\omega}\sin \omega t$
14	$\frac{s}{s^2 + \omega^2}$	$\cos \omega t$
15	$\frac{1}{s^2 - a^2}$	$\frac{1}{a}\sinh at$
16	$\frac{s}{s^2 - a^2}$	$\cosh at$
17	$\frac{1}{(s - a)^2 + \omega^2}$	$\frac{1}{\omega}e^{at}\sin \omega t$
18	$\frac{s - a}{(s - a)^2 + \omega^2}$	$e^{at}\cos \omega t$
19	$\frac{1}{s(s^2 + \omega^2)}$	$\frac{1}{\omega^2}(1 - \cos \omega t)$
20	$\frac{1}{s^2(s^2 + \omega^2)}$	$\frac{1}{\omega^3}(\omega t - \sin \omega t)$
21	$\frac{1}{(s^2 + \omega^2)^2}$	$\frac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$

Table of Laplace Transforms (*continued*)

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$
22	$\frac{s}{(s^2 + \omega^2)^2}$	$\frac{t}{2\omega}\sin \omega t$
23	$\frac{s^2}{(s^2 + \omega^2)^2}$	$\frac{1}{2\omega}(\sin \omega t + \omega t \cos \omega t)$
24	$\frac{s}{(s^2 + a^2)(s^2 + b^2)} \quad (a^2 \neq b^2)$	$\frac{1}{b^2 - a^2}(\cos at - \cos bt)$
25	$\frac{1}{s^4 + 4k^4}$	$\frac{1}{4k^3}(\sin kt \cos kt - \cos kt \sinh kt)$
26	$\frac{s}{s^4 + 4k^4}$	$\frac{1}{2k^2}\sin kt \sinh kt$
27	$\frac{1}{s^4 - k^4}$	$\frac{1}{2k^3}(\sinh kt - \sin kt)$
28	$\frac{s}{s^4 - k^4}$	$\frac{1}{2k^2}(\cosh kt - \cos kt)$
29	$\sqrt{s - a} - \sqrt{s - b}$	$\frac{1}{2\sqrt{\pi t^3}}(e^{bt} - e^{at})$
30	$\frac{1}{\sqrt{s + a}\sqrt{s + b}}$	$e^{-(a+b)t/2}I_0\left(\frac{a - b}{2}t\right)$
31	$\frac{1}{\sqrt{s^2 + a^2}}$	$J_0(at)$
32	$\frac{s}{(s - a)^{3/2}}$	$\frac{1}{\sqrt{\pi t}}e^{at}(1 + 2at)$
33	$\frac{1}{(s^2 - a^2)^k} \quad (k > 0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-1/2}I_{k-1/2}(at)$
34	e^{-as}/s	$u(t - a)$
35	e^{-as}	$\delta(t - a)$
36	$\frac{1}{s}e^{-k/s}$	$J_0(2\sqrt{kt})$
37	$\frac{1}{\sqrt{s}}e^{-k/s}$	$\frac{1}{\sqrt{\pi t}}\cos 2\sqrt{kt}$
38	$\frac{1}{s^{3/2}}e^{k/s}$	$\frac{1}{\sqrt{\pi k}}\sinh 2\sqrt{kt}$
39	$e^{-k\sqrt{s}} \quad (k > 0)$	$\frac{k}{2\sqrt{\pi t^3}}e^{-k^2/4t}$