Tfy-44.140 Quantum Computing (Kvanttilaskenta) Final Examination 7.1.2004

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1. Briefly explain the following basic concepts: a) Bell states. b) Quantum key distribution BB84 protocol. c) EPR paradox. d) Dense coding. e) Quantum teleportation. f) No-cloning theorem.

2. Dual vector space and projection operators.
a) Let

$$|x\rangle = \begin{pmatrix} 1 \\ i \\ 2+i \end{pmatrix}, \quad |y\rangle = \begin{pmatrix} 2-i \\ 1 \\ 2+i \end{pmatrix}.$$

Find $\|x\|$, $\langle x|y\rangle$ and $\langle y|x\rangle$.

b) Prove that for general $|x\rangle$, $|y\rangle$

$$\langle x|y\rangle = \langle y|x\rangle^*.$$

c) Let

$$|e_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, |e_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and

$$|v\rangle = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \sum c_k|e_k\rangle.$$

Find the coefficients c_1 and c_2. Employ projection operators.

3. Entanglement of states ?
a) Which of the following states are entangled states? If the state is a tensor product, decompose it to the product form.

$$(1) \quad \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$
$$(2) \quad \frac{1}{\sqrt{3}}(|000\rangle + |100\rangle + |101\rangle)$$
$$(3) \quad \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$
$$(4) \quad \frac{1}{\sqrt{2}}(|101\rangle + |111\rangle)$$
$$(5) \quad \frac{1}{2}(|000\rangle + |001\rangle + |010\rangle + |011\rangle)$$
$$(6) \quad \frac{1}{2}(|000\rangle + |010\rangle + |011\rangle + |111\rangle)$$
$$(7) \quad \frac{1}{2}(|000\rangle + |001\rangle + |100\rangle + |101\rangle)$$

b) Let $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Find the expectation value of $\sigma_x \otimes \sigma_z$ measured in this state.

4. Quantum gates.

(a) CNOT gate. Consider the Hamiltonian:

$$H = \hbar\omega \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

i) Write $U(t) = \exp(-iHt/\hbar)$ as a 4×4 matrix.

ii) Find the smallest time $t > 0$ for which $U(t) = \text{CNOT}$.

(b) Hadamard gate.

i) Show that the Walsh-Hadamard transformation $W_n = (H \otimes H \otimes \cdots \otimes H)$ is unitary.

ii) Let x be an n-bit binary number, $x \in \{0,1\}^n$. Show that

$$W_n|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y}|y\rangle,$$

where $x \cdot y \equiv \sum_{i=0}^{n-1} x_i y_i \pmod 2$.

iii) Show that the control and target bits are interchangable by introducing four Hadamard gates:

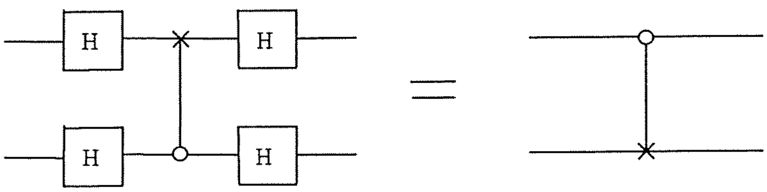
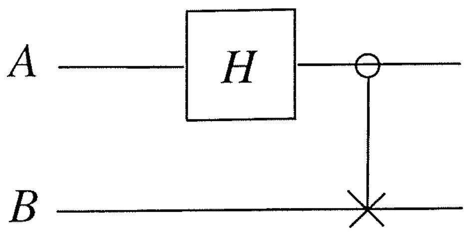


Fig: Control bit $\leftrightarrow$ Target bit.

iv) Consider the "quantum circuit" where A denotes the first qubit and B the second one:



(1)

What are the outputs for the inputs $|00\rangle, |01\rangle, |10\rangle$ and $|11\rangle$?

5. Quantum algorithms. Explain in brief:

a) Quantum Fourier transformation (QFT).

b) Shor's factorisation algorithm.

c) Grover's search algorithm.