$$\text{Nollalämpötilassa} \quad G^0_{\alpha\beta}(k,\omega) = \delta_{\alpha,\beta}\left[\frac{\theta(k-k_F)}{\omega-\omega_k+\mathrm{i}\eta} + \frac{\theta(k_F-k)}{\omega-\omega_k-\mathrm{i}\eta}\right] \tag{1}$$

1. Lausu vuorovaikuttamattoman ja homogeenisen vapaan kentän Greenin funktio

$$G^0_{\alpha\beta}(xt, x't') \equiv \langle\Phi_0|T[\hat{\psi}_\alpha(xt)\hat{\psi}^\dagger_\beta(x't')]|\Phi_0\rangle$$

integraalimuodossa Fermi-funktion avulla (käytä liikemääräesitystä $\hat{\psi}_\alpha(xt) = \frac{1}{\sqrt{V}}\sum_k \mathrm{e}^{\mathrm{i}(kx-\omega t)}\hat{a}_{k\alpha}$) ja muunna se sitten liikemäärä-energia-esitykseen

$$G^0_{\alpha\beta}(k,\omega) \equiv \frac{1}{(2\pi)^4}\int \mathrm{d}^3(x-x')\,\mathrm{d}(t-t')\,\mathrm{e}^{-\mathrm{i}[k\cdot(x-x')-\omega_k(t-t')]}G^0_{\alpha\beta}(xt,x't').$$

Johda näiden avulla

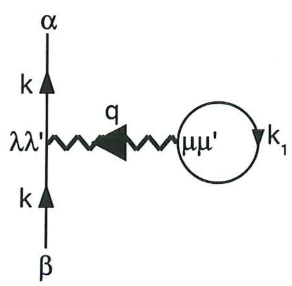
a) Greenin funktio $G^0_{\alpha\beta}(xt,x't')$ tyhjässä avaruudessa, missä ei ole muita elektroneja läsnä (laske integraalilauseke auki).

b) ylläoleva (yhtälö 1) nollalämpötilan Greenin funktio $G^0_{\alpha\beta}(k,\omega)$, kun Fermi-energia on ϵ_F. Tulos koostuu kahdesta termistä; miten ne tulkitaan?

Vinkkejä:

$$\int_{-\infty}^{\infty}\mathrm{e}^{-ax^2+2bx}\mathrm{d}x = \sqrt{\frac{\pi}{a}}\mathrm{e}^{b^2/a} \quad (\mathrm{Re}[a]\geq 0) \qquad \int_0^\infty \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t \to \lim_{\eta\to 0^+}\int_0^\infty \mathrm{e}^{-(\mathrm{i}\omega+\eta)t}\mathrm{d}t$$

2. Laske oheista diagrammia vastaava ensimmäisen kertaluokan korjaustermi Greenin funktioon $G_{\alpha\beta}(k,\omega)$. (Oleta vuorovaikutus spinriippumattomaksi.)



3. Suprajohtava BSC-perustila on muotoa

$$|BCS\rangle = \left[\prod_{\alpha>0}\left(u_\alpha + v_\alpha\hat{a}^\dagger_\alpha\hat{a}^\dagger_{-\alpha}\right)\right]|0\rangle.$$

- Mitä tarkoitetaan ehdolla $\alpha > 0$ ja symbolilla $-\alpha$?
- Millä ehdolla tila on oikein normitettu?
- Millä ehdolla siirtyminen operaattoreista $\hat{a}_\alpha$ operaattoreihin $\hat{b}_\alpha = u_\alpha\hat{a}_\alpha - v_\alpha\hat{a}^\dagger_{-\alpha} \Leftrightarrow \hat{a}_\alpha = u_\alpha\hat{b}_\alpha + v_\alpha\hat{b}^\dagger_{-\alpha}$ on kanoninen muunnos?
- Osoita, että $\hat{b}_\alpha|BCS\rangle = 0$
- Laske Wickin teoreemassa tarvittavat kontraktiot $\langle\hat{x}\hat{y}\rangle_{\mathrm{BSC}} \equiv \langle BCS|\hat{x}\hat{y}|BCS\rangle$ seuraaville operaattoripareille: $\hat{b}_\alpha\hat{b}_\beta$, $\hat{b}^\dagger_\alpha\hat{b}_\beta$, $\hat{b}_\alpha\hat{b}^\dagger_\beta$, $\hat{b}^\dagger_\alpha\hat{b}^\dagger_\beta$, ja $\hat{a}_\alpha\hat{a}_\beta$, $\hat{a}^\dagger_\alpha\hat{a}_\beta$, $\hat{a}_\alpha\hat{a}^\dagger_\beta$, $\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta$.

4. Mitä tarkoitetaan käsitteillä itseisenergia (self energy, Σ) ja aito itseisenergia (proper self energy, $\Sigma^\star$)? Piirrä diagrammien avulla ja selitä; kirjoita Greenin funktio $G_{\alpha\beta}(k,\omega)$ lausuttuna vapaan Greenin funktion ja aidon itseisenergian avulla, johda Dysonin yhtälö ja ratkaise. Osoita, että aito itseisenergia vastaa siirtymää liikemäärä-energia relaatiossa.

Tfy-44.139 Quantum mechanics IV Course examination 26 October, 2005

Janne Salo

$$\text{In zero temperature}\quad G^0_{\alpha\beta}(k,\omega)=\delta_{\alpha,\beta}\left[\frac{\theta(k-k_F)}{\omega-\omega_k+\mathrm{i}\eta}+\frac{\theta(k_F-k)}{\omega-\omega_k-\mathrm{i}\eta}\right] \tag{2}$$

1. Write the Green's function of a noninteracting and homogeneous free field,

$$G^0_{\alpha\beta}(xt,x't')\equiv\langle\Phi_0|T[\hat{\psi}_\alpha(xt)\hat{\psi}^\dagger_\beta(x't')]|\Phi_0\rangle$$

in integral form in terms of Fermi function (use the momentum basis, $\hat{\psi}_\alpha(xt)=\frac{1}{\sqrt{V}}\sum_k \mathrm{e}^{\mathrm{i}(kx-\omega t)}\hat{a}_{k\alpha}$) and then derive the momentum-energy presentation

$$G^0_{\alpha\beta}(k,\omega)\equiv\frac{1}{(2\pi)^4}\int \mathrm{d}^3(x-x')\,\mathrm{d}(t-t')\,\mathrm{e}^{-\mathrm{i}[k\cdot(x-x')-\omega_k(t-t')]}G^0_{\alpha\beta}(xt,x't').$$

Use these results to derive

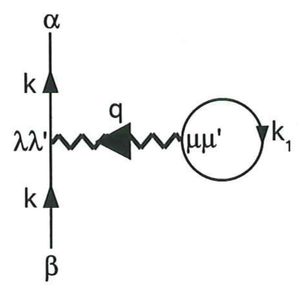
a) Green's function $G^0_{\alpha\beta}(xt,x't')$ in an empty space in absence of other electrons (solve the integral expression).

b) $G^0_{\alpha\beta}(k,\omega)$ (given by eq. 2 above) in zero temperature, with the Fermi energy ϵ_F. This will consist of two terms; how are they interpreted?

Hints:

$$\int_{-\infty}^{\infty}\mathrm{e}^{-ax^2+2bx}\mathrm{d}x=\sqrt{\frac{\pi}{a}}\mathrm{e}^{b^2/a}\quad(\mathrm{Re}[a]\geq 0)\qquad\qquad \int_0^\infty \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t\longrightarrow\lim_{\eta\to 0+}\int_0^\infty \mathrm{e}^{-(\mathrm{i}\omega+\eta)t}\mathrm{d}t$$

2. Calculate the first-order correction term to the Green's function $G_{\alpha\beta}(k,\omega)$, corresponding to the following diagram. (Assume spin-independent interaction.)



3. Superconducting BSC ground state is given by

$$|BCS\rangle=\left[\prod_{\alpha>0}\left(u_\alpha+v_\alpha\hat{a}^\dagger_\alpha\hat{a}^\dagger_{-\alpha}\right)\right]|0\rangle.$$

- What do the condition $\alpha>0$ and the symbolic notation $-\alpha$ mean?
- What is the condition for the state to be properly normalised?
- When does the following change of 'variables' represent a canonical transformation: $\hat{b}_\alpha=u_\alpha\hat{a}_\alpha-v_\alpha\hat{a}^\dagger_{-\alpha}\Leftrightarrow\hat{a}_\alpha=u_\alpha\hat{b}_\alpha+v_\alpha\hat{b}^\dagger_{-\alpha}$?
- Show that $\hat{b}_\alpha|BCS\rangle=0$
- Calculate the following contractions $\langle\hat{x}\hat{y}\rangle_{\mathrm{BSC}}\equiv\langle BCS|\hat{x}\hat{y}|BCS\rangle$, which are needed for the use of Wick's theorem: $\hat{b}_\alpha\hat{b}_\beta$, $\hat{b}^\dagger_\alpha\hat{b}_\beta$, $\hat{b}_\alpha\hat{b}^\dagger_\beta$, $\hat{b}^\dagger_\alpha\hat{b}^\dagger_\beta$, ja $\hat{a}_\alpha\hat{a}_\beta$, $\hat{a}^\dagger_\alpha\hat{a}_\beta$, $\hat{a}_\alpha\hat{a}^\dagger_\beta$, $\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta$.

4. What is meant with the concepts *self energy*, Σ, and *proper self energy*, Σ^*? Explain using diagrams, write the Green's function $G_{\alpha\beta}(k,\omega)$ in terms of the free Green's function and the proper self energy, derive the Dyson's equation and solve. Show that the proper self energy corresponds to a shift in the momentum-energy relation.