K1 I välikoe

1. a) $z = \frac{2+i}{3-i} = \frac{(2+i)(3+i)}{(3-i)(3+i)} = \frac{6+5i+i^2}{3^2-i^2} = \frac{5+5i}{10} = \frac{1}{2} + \frac{1}{2}i$

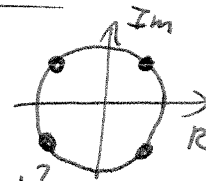
$\bar{z} = \frac{1}{2} - \frac{1}{2}i$

b) [Adams. esim. A.I.8, luennollakin] $\quad -4 = 4e^{i\pi} = 4 \cdot e^{i\pi + 2\pi k i} \quad \forall k \in \mathbb{Z}$

$z^4 = 4, \quad z =: Re^{i\varphi} \Rightarrow z^4 = R^4 e^{i4\varphi} = 4e^{i(\pi+2\pi k)}$

$\Rightarrow \begin{cases} R^4 = 4 \quad (\text{ja } R \in \mathbb{R},\ R \geq 0) \\ 4\varphi = \pi + 2\pi k,\ k \in \mathbb{Z} \end{cases} \Rightarrow \begin{cases} R = \sqrt{2} \\ \varphi = \frac{\pi}{4} + \frac{\pi}{2}k,\ k = 0,1,2,3. \end{cases}$

4 ratkaisua: $\sqrt{2}\, e^{i\varphi}$ jossa $\underbrace{\varphi}_{\text{ja } k}$ yo. muotoa.



Voi myös sieventää muotoon $z = \{1+i, -1+i, -1-i, 1-i\}$

Ratkaisut symmetrisin välein $\sqrt{2}$-säteiselle ympyrälle. ☐.

Onnistuisi myös näin: $z = a+bi;\ a, b \in \mathbb{R}.\quad z^4 = ((a+bi)^2)^2 = -4$

eli $(a^2 + (bi)^2 + i2ab)^2 = -4$

$(a^2-b^2)^2 + (i2ab)^2 + i4ab(a^2-b^2) = -4$

$a^4 + b^4 - 6a^2b^2 + i \cdot 4ab(a+b)(a-b) = -4 + i \cdot 0$

$\begin{cases} \text{Re-osat: } a^4 + b^4 - 6a^2b^2 = -4 \\ \text{Im-osat: } 4ab(a+b)(a-b) = 0 \end{cases}$ $\Rightarrow$ 4 tapausta: $a=0$ tai $b=0$ tai $a=-b$ tai $a=b$.

$\underline{a=0} \Rightarrow b^4 = -4$ ei onnistu, $b \in \mathbb{R}$.

$\underline{b=0} \Rightarrow a^4 = -4$ — " — $a \in \mathbb{R}$.

$\underline{a=-b} \Rightarrow b^4 + b^4 - 6b^2b^2 = -4$ eli $b^4 = 1$ siis $b = \pm 1$, $a = -b$

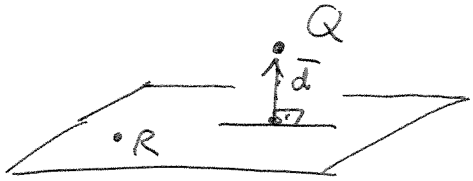
eli ratkaisut $-1+i,\ 1-i$ (2 kpl)

$\underline{a=b} \Rightarrow b^4 + b^4 - 6b^2b^2 = -4$ eli $b^4 = 1$, taas $b = \pm 1$ mutta $a = b$.

Eli ratkaisut $1+i,\ -1-i$ (2 kpl lisää)

☐

2. a)



$Q = \bar{i} + 2\bar{j} = (1,2,0)$

$\bar{n} = (3,-4,-5)$

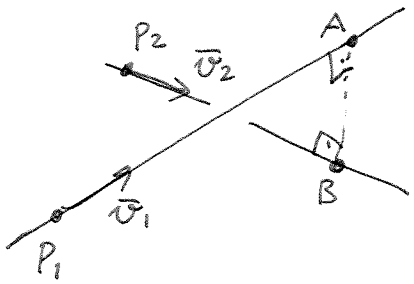
R = piste tasolta, etsi. Ota vaikkapa $x = 1$, $y = 2$

ja sijoita $\Rightarrow$ $3 \cdot 1 - 4 \cdot 2 - 5z = 2 \Rightarrow z = \frac{-7}{5}$ eli $R = (1, 2, -\frac{7}{5})$

$\bar{d} = \overrightarrow{RQ}$:n projektio $\bar{n}$:lle ja $|\bar{d}| = \frac{|\overrightarrow{RQ} \cdot \bar{n}|}{|\bar{n}|}$ = etäisyys.

$\overrightarrow{RQ} = (0, 0, \frac{7}{5})$, $|\bar{n}| = 50$ $\Rightarrow$ $|\bar{d}| = \frac{7}{\cancel{5}\sqrt{50}}$.

b)



$\overrightarrow{AB} = \overrightarrow{P_1P_2}$:n projektio $\bar{v}_1 \times \bar{v}_2$:lle

$P_1 = (1,2,3)$ $\bar{v}_1 = (2,-3,-4)$

$P_2 = (-1,0,1)$ $\bar{v}_2 = (2,-1,7)$

$|\overrightarrow{AB}| = \frac{|\overrightarrow{P_1P_2} \cdot (\bar{v}_1 \times \bar{v}_2)|}{|\bar{v}_1 \times \bar{v}_2|}$ = etäisyys $\overrightarrow{P_1P_2} = (-2,-2,-2)$

$\bar{v}_1 \times \bar{v}_2 = \bar{i}\begin{vmatrix} -3 & -4 \\ -1 & 7 \end{vmatrix} - \bar{j}\begin{vmatrix} 2 & -4 \\ 2 & 7 \end{vmatrix} + \bar{k}\begin{vmatrix} 2 & -3 \\ 2 & -1 \end{vmatrix} = (-25, -22, 4)$

$|\overrightarrow{AB}| = \frac{86}{\sqrt{1125}}$

(3) gaussita:

$$\begin{array}{cccc|c} & -5 & -15 & -2\alpha-5 & \beta+5 \\ & 2 & 5 & 3 & 0 \\ \cdot(-2) \quad \cdot 3 & -3 & -9 & \alpha+8 & 6 \\ & 1 & 3 & -1 & -2 \end{array}$$

1-sarakkeesta: "1" tukialkioksi

$$\Rightarrow \begin{array}{cccc|c} 1 & 3 & -1 & -2 \\ 0 & -1 & 5 & 4 \\ 0 & 0 & \alpha+5 & 0 \\ -5 & -15 & -2\alpha-5 & \beta+5 \end{array} \cdot 5 \Rightarrow \begin{array}{ccc|c} 1 & 3 & -1 & -2 \\ 0 & -1 & 5 & 4 \\ 0 & 0 & \alpha+5 & 0 \\ 0 & 0 & -2\alpha-10 & \beta-5 \end{array} \cdot 2$$

$$\Rightarrow \begin{array}{ccc|c} 1 & 3 & -1 & -2 \\ 0 & -1 & 5 & 4 \\ 0 & 0 & \alpha+5 & 0 \\ 0 & 0 & 0 & \beta-5 \end{array}$$

Jos $\beta \neq 5$ ei ratkaisua ts. 0 kpl

Jos $\boxed{\beta = 5:}$ $\begin{array}{ccc|c} 1 & 3 & -1 & -2 \\ & -1 & 5 & 4 \\ & & \alpha+5 & 0 \end{array}$

jolloin jos $\alpha \neq -5 \Rightarrow$ kääntyvä kolmio joten 1 ratk

jos $\alpha = -5$, vapaa parametri eli ∞ ratkaisua.

(4.) a) $T \Leftrightarrow A = \begin{pmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{pmatrix}$ eli $T(\bar{x}) = A\bar{x} \quad \forall \bar{x} \in \mathbb{R}^3$.

b) surjektio? $A\bar{x} = \bar{b} \iff \begin{array}{ccc|c} 1 & -5 & 4 & b_1 \\ 0 & 1 & -6 & b_2 \end{array} \cdot 5 \Rightarrow \begin{array}{ccc|c} 1 & 0 & -26 & b_1+5b_2 \\ 0 & 1 & -6 & b_2 \end{array}$

x_3 mielivalt., olkoon $x_3 = 0 \Rightarrow \begin{cases} x_1 = b_1 + 5b_2 \\ x_2 = b_2 \end{cases}$ kelpaa ratkaisuksi

eli $\exists \bar{x}$ s.e. $T(\bar{x}) = \bar{b}$ joten on surjektio. $\boxtimes$.

c) injektio? $\iff A\bar{x} = \bar{0}$:lla ainoa ratk. $\bar{x} = \bar{0}$. b):stä: $A\bar{x} = \bar{0} \iff \begin{array}{ccc|c} 1 & 0 & -26 & 0 \\ 0 & 1 & -6 & 0 \end{array}$

Tässä on vapaa parametri (x_3) joten ei ole injektio. $\boxtimes$.

(Esim. $\bar{x} = (26, 6, 1)$ ja $\bar{y} = (52, 12, 2)$ antavat $T(\bar{x}) = T(\bar{y}) = \bar{0}$.)