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Mat-1.3460 Principles of Functional Analysis

Mid-term exam (2/2), 16.12.2006

1. Let X be a Banach space and $T \in B(X)$. Assume that $\|T^k\| \leq c$, $k = 0, 1, 2, \ldots$. Prove that for every $N \in \mathbf{N}$, $\sigma(T^N) \subset \{\lambda \in \mathbf{C} \ : \ |\lambda| \leq 1\}$, and

$$\|(\lambda I - T^N)^{-1}\| \leq \frac{c}{|\lambda| - 1}$$

for $\lambda \in \mathbf{C}$ with $|\lambda| > 1$.

2. For $H = \ell^2$, define the shift operators S_R and S_L by

$$S_R x = (0, \xi_1, \xi_2, \ldots),$$
$$S_L x = (\xi_2, \xi_3, \ldots),$$

where $x = (\xi_1, \xi_2, \ldots) \in \ell^2$. Show that

(a) $\sigma_p(S_R) = \emptyset$ and $\sigma_p(S_L) = \{\lambda \in \mathbf{C} : |\lambda| < 1\} =: \mathbb{D}$,

(b) $\sigma(S_R) = \sigma(S_L) = \{\lambda \in \mathbf{C} : |\lambda| \leq 1\}$,

(c) Find some approximate eigenvectors for S_R in order to show that $\partial\mathbb{D} \subset \sigma_a(S_R)$.

3. Let T be a linear operator in Banach space ℓ^1 defined by

$$Te_n := \alpha_n(e_n + e_{n+1} + \cdots + e_{2n})$$

for all $n \in \mathbf{N}$, where $(e_n)_{n=1}^{\infty}$ is the orthonormal basis, i.e., $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, 0, \ldots)$, etc. Give the necessary and sufficient conditions for the sequence $(\alpha_n)_{n=1}^{\infty}$, where $\alpha_n \in \mathbf{C}$, so that

(a) $T \in B(\ell^1)$,

(b) $T \in \mathcal{K}(\ell^1)$,

(c) $T \in \mathcal{F}(\ell^1)$.

Justify your answer.